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Common Fixed Point Theorems in Fuzzy Normed Space

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Abstract

Fixed point theory is an intriguing topic with numerous applications in several branches of mathematics. In addition, fixed point theory offers useful methods for problem-solving that may be used across many subfields of mathematical analysis. Fixed point theorems are concerned with mappings f of a set X into itself that, under particular conditions, permit a fixed point, that is, a point $x \in X$ such that f(x) = x. In this paper, we use the triangle property on fuzzy normed space(Fn-space) to show a common fixed point (CF-point) without continuity. First, we review some of the fundamental terms used in the fuzzy context. After that the notion of triangle fuzzy norm is given then we show that the self mappings Γ and Λ have a CF-point in the setting of Fn-space. Additionally, certain applications of our main results to the Fredholm integral equation are investigated.

Introduction

One of the most significant domains in present-day mathematics is functional analysis. It is fundamental to the theory of differential equations, representation theory, probability, and the investigation of a variety of distinctive properties of a variety of spaces. see[1-4].

Schweizer and Sklar [5] established the definition of a continuous triangular norm in 1960. Zadeh [6] then presented the notion of fuzzy sets in 1965 and numerous mathematicians have studied this concept and obtained different main results from various points of view. Authors in [7] produced the fuzzy metric space using the idea of fuzziness and the continuous t-norm. There have been numerous works published about fuzzy metric spaces; for examples, see references [8–11]. In a linear space, Katsaras introduced the fuzzy norm in 1984. Numerous studies have been conducted on Fn-spaces; see [12–16].

On the other hand, fixed point theory provides a straightforward, effective, and practical method for nonlinear analysis. Additionally, it has beneficial applications in the disciplines of mathematics and other sciences (see [17-20]). As a consequence, this theory has drawn a

significant number of researchers who are leading the theory's development in a variety of fields.

In this paper, we employ the triangle property on fuzzy norms to show a CF-point in the context of Fn-space. In addition, certain applications of our key conclusions to the Fredholm integral equation are studied. Structurally, this paper involves the following: we begin with the fundamental terms utilized in the fuzzy context and the definition of fuzzy normed space. after that, we introduce the fuzzy norm's triangle property. This property is used to prove a CF-point theorem for contractive mappings in a Fn-space. In addition, we examine the existence and unique solution to Fredholm integral. Finally, the paper finished with a conclusion section.

Preliminaries

In this part, we will go over certain terminology and results that will be used throughout the rest of the research. To begin, let's review some of the fundamental terms utilized in the fuzzy context.

Definition 1[5]. A binary operation \circledast : $[0,1] \times [0,1] \to [0,1]$ is called t-norm if it fulfills the requirements below for all $\mathfrak{s}, r, t, \varsigma \in [0,1]$,

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(i) 1 \circledast r = r,

(ii) r \circledast \varsigma = \varsigma \circledast r,

(iii) r \circledast (t \circledast \varsigma) = (r \circledast t) \circledast \varsigma

(iv) If r \le \varsigma and t \le \varsigma then r \circledast t \le \varsigma \circledast \varsigma.
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Definition 2 [21]. Let L be a non-empty vector space over the field \mathbb{F} (\mathbb{C} *or* \mathbb{R}). A Fn-space is represented by the triplet (L, $\widetilde{\mathbb{N}}$, \circledast), where \circledast is a continuous t-norm and $\widetilde{\mathbb{N}}$ signify a fuzzy set on $L \times \mathbb{R}$ fulfilling the requirements for each x, y \in L:

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\begin{array}{ll} (1)\widetilde{\mathrm{N}}(\mathrm{x},0) &= 0,\\ (2)\ \widetilde{\mathrm{N}}(\mathrm{x},\tau) &= 1, \forall \tau > 0 \ \text{if and only if} \ \mathrm{x} = 0,\\ (3)\ \widetilde{\mathrm{N}}(\mathrm{rx},\tau) &= \widetilde{\mathrm{N}}(\mathrm{x},\tau/|\mathrm{r}|), \ \text{for each} \ 0 \neq \mathrm{r} \in \mathbb{R}, \tau \geq 0\\ (4)\ \widetilde{\mathrm{N}}(\mathrm{x},\tau) &\circledast \widetilde{\mathrm{N}}(y,s) &\leq \widetilde{\mathrm{N}}(\mathrm{x}+y,\tau+s), \ \forall \tau,s \geq 0\\ (5)\ \widetilde{\mathrm{N}}(\mathrm{x},\cdot) \ \text{is left continuous for all} \ \mathrm{x} \in L, \ \text{and} \ \lim_{\tau \to \infty} \widetilde{\mathrm{N}}(\mathrm{x},\tau) &= 1. \end{array}
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Definition 3[22]. Let $(L, \widetilde{N}, \circledast)$ be a Fn-space. Then

- (1) a sequence $\{p_n\}$ is termed as a convergent sequence if $\lim_{\tau \to \infty} \widetilde{N}(p_n p, \tau) = 1$ for each $\tau > 0$ and $p \in L$.
- (2) a sequence $\{p_n\}$ is termed as a Cauchy if $\lim_{n\to\infty}\widetilde{N}\big(p_{n+j}-p_n,\tau\big)=1$; for each $\tau>0$ and j=1,2,...

Definition 4[22]. Let $(L, \widetilde{N}, \circledast)$ be a Fn-space. Then $(L, \widetilde{N}, \circledast)$ is termed as complete if every Cauchy sequence in L is convergent in L.

Main Results:

This section introduces the fuzzy norm's triangle property. This property is used to prove a CF-point theorem for contractive mappings in a Fn-space.

A point $x \in L$ is said to be a CF-point if a pair of self mappings (Γ, Λ) on L satisfy $\Gamma x = \Lambda x = x$.

Definition 5. Let $(L, \tilde{N}, \circledast)$ be a Fn-space. A fuzzy norm \tilde{N} is called triangular if the condition holds:

$$\frac{1}{\widetilde{N}(\mathbf{x} - \boldsymbol{y}, \tau)} - 1 \le \left(\frac{1}{\widetilde{N}(\mathbf{x} - \boldsymbol{z}, \tau)} - 1\right) + \left(\frac{1}{\widetilde{N}(\boldsymbol{y} - \boldsymbol{z}, \tau)} - 1\right)$$

for every $x, y, z \in L$ and $\tau > 0$

Theorem 1. Suppose that $(L, \widetilde{N}, \circledast)$ is fuzzy Banach space and \widetilde{N} is triangular. Let $\Gamma, \Lambda: L \to L$ be self-mappings with

$$\begin{split} \frac{1}{\tilde{N}(\Gamma\mathbf{x}-\Lambda\boldsymbol{y},\tau)} - 1 &\leq \lambda_1 \left(\frac{1}{\tilde{N}(\mathbf{x}-\boldsymbol{y},\tau)} - 1 \right) + \lambda_2 \left(\frac{\tilde{N}(\mathbf{x}-\boldsymbol{y},\tau)}{\tilde{N}(\mathbf{x}-\Lambda\boldsymbol{y},2\tau) \odot \tilde{N}(\boldsymbol{y}-\Gamma\mathbf{x},2\tau)} - 1 \right) \\ &+ \lambda_3 \left(\frac{\tilde{N}(\mathbf{x}-\Gamma\mathbf{x},\tau) \odot \tilde{N}(\boldsymbol{y}-\Lambda\boldsymbol{y},\tau)}{\tilde{N}(\mathbf{x}-\boldsymbol{y},\tau) \odot \tilde{N}(\boldsymbol{y}-\Lambda\boldsymbol{y},2\tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\tilde{N}(\mathbf{x}-\Gamma\mathbf{x},\tau)} - 1 + \frac{1}{\tilde{N}(\boldsymbol{y}-\Lambda\boldsymbol{y},\tau)} - 1 \right) & \dots (1) \end{split}$$

for all x, $y \in L$, $\tau > 0$, $\lambda_1 \in (0,1)$ and λ_2 , λ_3 , $\lambda_4 \ge 0$ with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$. Then Γ and Λ have a CF-point in L.

Proof: Fix $x_0 \in L$ and generate a sequence of points in L,

$$\mathbf{x}_{2j+1} = \Gamma \mathbf{x}_{2j}$$

 $\mathbf{x}_{2j+2} = \Lambda \mathbf{x}_{2j+1}$, $j \ge 0$

Then,

$$\begin{split} \frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 &= \frac{1}{\widetilde{N}(\Gamma \mathbf{x}_{2j} - \Lambda \mathbf{x}_{2j+1}, \tau)} - 1 \\ &\leq \lambda_1 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \Lambda \mathbf{x}_{2j+1}, \tau)} - 1 \right) \\ &+ \lambda_3 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \Gamma \mathbf{x}_{2j}, \tau) \circledast \widetilde{N}(\mathbf{x}_{2j+1} - \Lambda \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau) \circledast \widetilde{N}(\mathbf{x}_{2j} - \Lambda \mathbf{x}_{2j+1}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \Gamma \mathbf{x}_{2j}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \Lambda \mathbf{x}_{2j+1}, \tau)} - 1 \right) \\ &= \lambda_1 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \Gamma \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, 2\tau) \otimes \widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_3 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, 2\tau) \otimes \widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &= \lambda_1 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 \right) \\ &+ \lambda_2 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}$$

Since $\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+2}, 2\tau) \geq \widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau) \circledast \widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)$ for $\tau > 0$, we have further $\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \leq \lambda_1 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1\right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau) \circledast \widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1\right) + \lambda_3 \left(\frac{\widetilde{N}(\mathbf{x}_{2j-1} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j}, \tau) \circledast \widetilde{N}(\mathbf{x}_{2j-1} - \mathbf{x}_{2j+2}, \tau)} - 1\right) + \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1\right)$

Following simplification, we arrive at:

$$\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1}-\mathbf{x}_{2j+2},\tau)} - 1 \le \vartheta\left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j}-\mathbf{x}_{2j+1},\tau)} - 1\right) \qquad ...(2)$$

$$where \, \vartheta = \frac{\lambda_1 + \lambda_3 + \lambda_4}{1 - \lambda_2 - \lambda_1} < 1, \, since \, \lambda_2, \lambda_3, \lambda_4 \ge 0 \, with \, \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$$

Similarly,

$$\frac{1}{\widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau)} - 1 = \frac{1}{\widetilde{N}(\Gamma \mathbf{x}_{2j+1} - \Lambda \mathbf{x}_{2j+2}, \tau)} - 1$$

$$\leq \lambda_{1} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) + \lambda_{2} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right)$$

$$+ \lambda_{3} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \Gamma \mathbf{x}_{2j+1}, \tau) \otimes \widetilde{N}(\mathbf{x}_{2j+2} - \Lambda \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau) \otimes \widetilde{N}(\mathbf{x}_{2j+1} - \Lambda \mathbf{x}_{2j+2}, 2\tau) \otimes \widetilde{N}(\mathbf{x}_{2j+2} - \Gamma \mathbf{x}_{2j+1}, 2\tau)} - 1 \right)$$

$$+ \lambda_{4} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \Gamma \mathbf{x}_{2j+1}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{x}_{2j+2} - \Lambda \mathbf{x}_{2j+2}, \tau)} - 1 \right)$$

$$= \lambda_{1} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) + \lambda_{2} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau) \otimes \widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+2}, \tau)} - 1 \right)$$

$$+ \lambda_{3} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau) \otimes \widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau)} - 1 \right)$$

$$+ \lambda_{4} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau)} - 1 \right)$$

$$= \lambda_{1} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) + \lambda_{2} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+3}, \tau)} - 1 \right) + \lambda_{3} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+3}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau)} - 1 \right)$$

$$+ \lambda_{4} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) + \lambda_{2} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+3}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+3}, \tau)} - 1 \right) + \lambda_{3} \left(\frac{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+3}, \tau)}{\widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau)} - 1 \right)$$

Since $\widetilde{N}(x_{2j+1} - x_{2j+3}, 2\tau) \ge \widetilde{N}(x_{2j+1} - x_{2j+2}, \tau) \circledast \widetilde{N}(x_{2j+2} - x_{2j+3}, \tau)$ for $\tau > 0$, we have further

$$\begin{split} &\frac{1}{\widetilde{N}\left(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau\right)} - 1 \leq \lambda_{1} \left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau\right)} - 1\right) \\ &+ \lambda_{2} \left(\frac{\widetilde{N}\left(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau\right)}{\widetilde{N}\left(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau\right) \circledast \widetilde{N}\left(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau\right)} - 1\right) \\ &+ \lambda_{3} \left(\frac{\widetilde{N}\left(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau\right)}{\widetilde{N}\left(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau\right)} - 1\right) \\ &+ \lambda_{4} \left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau\right)} - 1 + \frac{1}{\widetilde{N}\left(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau\right)} - 1\right) \end{split}$$

Following simplification, we arrive at:

$$\frac{1}{\tilde{N}(x_{2j+2} - x_{2j+3}, \tau)} - 1 \le \vartheta\left(\frac{1}{\tilde{N}(x_{2j+1} - x_{2j+2}, \tau)} - 1\right) \qquad ...(3)$$

for $\tau > 0$ where $\vartheta = \frac{\lambda_1 + \lambda_3 + \lambda_4}{1 - \lambda_2 - \lambda_4} < 1$, since $\lambda_2, \lambda_3, \lambda_4 \ge 0$ with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$

Now, by induction and using (2) and (3), we obtain

$$\frac{1}{\widetilde{N}(\mathbf{x}_{2j+2} - \mathbf{x}_{2j+3}, \tau)} - 1 \le \vartheta \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1\right)$$

$$\le \vartheta^2 \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1\right)$$

$$\vdots$$

$$\le \vartheta^{2j+2} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{0} - \mathbf{x}_{1}, \tau)} - 1\right)$$

Accordingly,

$$\lim_{j \to \infty} \widetilde{N}(x_{2j+1} - x_{2j+2}, \tau) = 1, \text{ for } \tau > 0.$$
 ...(4)

Because \widetilde{N} is triangular, it follows that we may conclude,

$$\begin{split} \frac{1}{\widetilde{N}\left(\mathbf{x}_{j}-\mathbf{x}_{k},\tau\right)}-1 &\leq \left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{j}-\mathbf{x}_{j+1},\tau\right)}-1\right)+\left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{j+1}-\mathbf{x}_{j+2},\tau\right)}-1\right) \\ &+\cdots\left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{k-1}-\mathbf{x}_{k},\tau\right)}-1\right) \\ &\leq \left(\vartheta^{j}+\vartheta^{j+1}+\cdots+\vartheta^{k-1}\right)\left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{\circ}-\mathbf{x}_{1},\tau\right)}-1\right) \\ &\leq \frac{\vartheta^{j}}{1-\vartheta^{j}}\left(\frac{1}{\widetilde{N}\left(\mathbf{x}_{\circ}-\mathbf{x}_{1},\tau\right)}-1\right)\to 0 \ as \ j\to 0 \end{split}$$

Consequently $\{x_i\}$ is cauchy sequence in L.

Now because L is complete, then $\psi_1 \in L$ exists with $\lim_{i \to \infty} \widetilde{N}(x_{2j+1} - \psi_1, \tau) = 1, \text{ for } \tau > 0. \qquad ...(5)$

Now, to show that $\Lambda y_1 = y_1$.

$$\frac{1}{\widetilde{N}(y_{1}-\Lambda y_{1},\tau)} - 1 \leq \left(\frac{1}{\widetilde{N}(y_{1}-x_{2j+1},\tau)} - 1\right) + \left(\frac{1}{\widetilde{N}(x_{2j+1}-\Lambda y_{1},\tau)} - 1\right) \qquad \dots(6)$$
for $\tau > 0$. By (1), (4), and (5) we have,
$$\left(\frac{1}{\widetilde{N}(x_{2j+1}-\Lambda y_{1},\tau)} - 1\right) = \frac{1}{\widetilde{N}(\Gamma x_{2j}-\Lambda y_{1},\tau)}$$

$$\leq \lambda_{1} \left(\frac{1}{\widetilde{N}(x_{2j}-y_{1},\tau)} - 1\right)$$

$$+\lambda_{2} \left(\frac{\widetilde{N}(x_{2j}-y_{1},\tau)}{\widetilde{N}(x_{2j}-\Lambda y_{1},2\tau) \otimes \widetilde{N}(y_{1}-\Gamma x_{2j},2\tau)} - 1\right)$$

$$+\lambda_{3} \left(\frac{\widetilde{N}(x_{2j}-\Gamma x_{2j},\tau) \otimes \widetilde{N}(y_{1}-\Lambda y_{1},\tau)}{\widetilde{N}(x_{2j}-\gamma y_{1},\tau) \otimes \widetilde{N}(x_{2j}-\gamma y_{1},\tau)} - 1\right)$$

$$+\lambda_{4} \left(\frac{1}{\widetilde{N}(x_{2j}-\Gamma x_{2j},\tau)} - 1 + \frac{1}{\widetilde{N}(y_{1}-\Lambda y_{1},\tau)} - 1\right)$$

$$= \lambda_{1} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau)} - 1 \right) \\ + \lambda_{2} \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{A}\mathbf{y}_{1}, 2\tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{x}_{2j+1}, 2\tau)} - 1 \right) \\ + \lambda_{3} \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, 2\tau)} - 1 \right) \\ + \lambda_{4} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)} - 1 \right) \\ Since \ \widetilde{N}(\mathbf{x}_{2j} - \mathbf{A}\mathbf{y}_{1}, 2\tau) \ge \widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau) \text{ for } \tau > 0 \text{ then}$$

$$\left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{A}\mathbf{y}_{1}, \tau)} - 1 \right) \le \lambda_{1} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau)} - 1 \right) \\ + \lambda_{2} \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)} - 1 \right) \\ + \lambda_{3} \left(\frac{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{y}_{1}, \tau) \odot \widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)} - 1 \right) \\ + \lambda_{4} \left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j} - \mathbf{x}_{2j+1}, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)} - 1 \right) \\ \rightarrow (\lambda_{2} + \lambda_{4}) \left(\frac{1}{\widetilde{N}(\mathbf{y}_{1} - \mathbf{A}\mathbf{y}_{1}, \tau)} - 1 \right) \text{ as } j \to \infty$$

Then,

$$\lim_{j\to\infty} \sup\left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \Lambda \mathbf{y}_{1}, \tau)} - 1\right) \leq (\lambda_{2} + \lambda_{4})\left(\frac{1}{\widetilde{N}(\mathbf{y}_{1} - \Lambda \mathbf{y}_{1}, \tau)} - 1\right)$$

By (5) and (6) obtain

$$\left(\frac{1}{\widetilde{N}(y_1 - \Lambda y_1, \tau)} - 1\right) \le (\lambda_2 + \lambda_4)\left(\frac{1}{\widetilde{N}(y_1 - \Lambda y_1, \tau)} - 1\right)$$

Note that $\lambda_2 + \lambda_4 < 1$ because $\lambda_2, \lambda_3, \lambda_4 \ge 0$ with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$. Then, $\widetilde{N}(y_1 - \Lambda y_1, \tau) = 1$

Therefore $\Lambda y_1 = y_1$. Likewise, we can show that $\Gamma y_1 = y_1$ Since \widetilde{N} is triangular,

$$\frac{1}{\widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1 \le \left(\frac{1}{\widetilde{N}(y_1 - x_{2j+2}, \tau)} - 1\right) + \left(\frac{1}{\widetilde{N}(x_{2j+2} - \Gamma y_1, \tau)} - 1\right) \qquad ...(7)$$

$$Bv(1).(4). \ and \ (5) \ for \ \tau > 0.$$

$$\begin{split} \frac{1}{\widetilde{N}(x_{2j+2} - \Gamma y_{1}, \tau)} - 1 \end{pmatrix} &= \left(\frac{1}{\widetilde{N}(\Gamma y_{1} - \Lambda x_{2j+1}, \tau)} - 1\right) \\ &\leq \lambda_{1} \left(\frac{1}{\widetilde{N}(y_{1} - x_{2j+1}, \tau)} - 1\right) \\ &+ \lambda_{2} \left(\frac{N(y_{1} - x_{2j+1}, \tau)}{\widetilde{N}(y_{1} - \Lambda x_{2j+1}, 2\tau) \odot \widetilde{N}(x_{2j+1} - \Gamma y_{1}, 2\tau)} - 1\right) \\ &+ \lambda_{3} \left(\frac{\widetilde{N}(y_{1} - \Gamma y_{1}, \tau) \odot \widetilde{N}(x_{2j+1} - \Lambda x_{2j+1}, \tau)}{\widetilde{N}(y_{1} - \Lambda x_{2j+1}, \tau) \odot \widetilde{N}(x_{2j+1} - \Lambda x_{2j+1}, \tau)} - 1\right) \\ &+ \lambda_{4} \left(\frac{1}{\widetilde{N}(y_{1} - \Gamma y_{1}, \tau)} - 1 + \frac{1}{\widetilde{N}(x_{2j+1} - \Lambda x_{2j+1}, \tau)} - 1\right) \\ &= \lambda_{1} \left(\frac{1}{\widetilde{N}(y_{1} - X_{2j+1}, \tau)} - 1\right) \end{split}$$

$$\begin{split} & + \lambda_{2} \left(\frac{\tilde{N}(y_{1} - \mathbf{x}_{2j+1}, \tau)}{\tilde{N}(y_{1} - \mathbf{x}_{2j+2}, 2\tau) \cdot \mathfrak{N}(\mathbf{x}_{2j+1} - \Gamma y_{1}, 2\tau)} - 1 \right) \\ & + \lambda_{3} \left(\frac{\tilde{N}(y_{1} - \Gamma y_{1}, \tau) \cdot \mathfrak{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)}{\tilde{N}(y_{1} - \mathbf{x}_{2j+1}, \tau) \cdot \mathfrak{N}(y_{1} - \mathbf{x}_{2j+2}, 2\tau) \cdot \mathfrak{N}(\mathbf{x}_{2j+1} - \Gamma y_{1}, 2\tau)} - 1 \right) \\ & + \lambda_{4} \left(\frac{1}{\tilde{N}(y_{1} - \Gamma y_{1}, \tau)} - 1 + \frac{1}{\tilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1 \right) \end{split}$$

Since
$$\widetilde{N}(\mathbf{x}_{2j+1} - \Gamma y_1, 2\tau) \geq \widetilde{N}(\mathbf{x}_{2j+1} - y_1, \tau) \otimes \widetilde{N}(y_1 - \Gamma y_1, \tau)$$
 then,
$$\left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+2} - \Gamma y_1, \tau)} - 1\right) \leq \lambda_1 \left(\frac{1}{\widetilde{N}(y_1 - \mathbf{x}_{2j+1}, \tau)} - 1\right)$$

$$+ \lambda_2 \left(\frac{\widetilde{N}(y_1 - \mathbf{x}_{2j+2}, 2\tau) \otimes \widetilde{N}(\mathbf{x}_{2j+1} - y_1, \tau) \otimes \widetilde{N}(y_1 - \Gamma y_1, \tau)}{\widetilde{N}(y_1 - \Gamma y_1, \tau) \otimes \widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1\right)$$

$$+ \lambda_3 \left(\frac{\widetilde{N}(y_1 - \mathbf{x}_{2j+1}, \tau) \otimes \widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)}{\widetilde{N}(y_1 - \mathbf{x}_{2j+1}, \tau) \otimes \widetilde{N}(y_1 - \mathbf{x}_{2j+2}, 2\tau)} \otimes \widetilde{N}(\mathbf{x}_{2j+1} - y_1, \tau) \otimes \widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1\right)$$

$$+ \lambda_4 \left(\frac{1}{\widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{x}_{2j+1} - \mathbf{x}_{2j+2}, \tau)} - 1\right)$$

$$\to (\lambda_2 + \lambda_4) \left(\frac{1}{\widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1\right) as j \to \infty$$

Then,

$$\lim_{j\to\infty} \sup\left(\frac{1}{\widetilde{N}(\mathbf{x}_{2j+2}-\Gamma y_1,\tau)}-1\right) \leq (\lambda_2+\lambda_4)\left(\frac{1}{\widetilde{N}(y_1-\Gamma y_1,\tau)}-1\right)$$

By (7) and (5) obtain

$$\left(\frac{1}{\widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1\right) \le (\lambda_2 + \lambda_4)\left(\frac{1}{\widetilde{N}(y_1 - \Gamma y_1, \tau)} - 1\right)$$

Note that $\lambda_2 + \lambda_4 < 1$ because $\lambda_2, \lambda_3, \lambda_4 \ge 0$ with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$. Then, $\widetilde{N}(y_1 - \Gamma y_1, \tau) = 1$

Therefore $\Gamma y_1 = y_1$. Consequently, y_1 is a CF-point of Γ and Λ .

Example 1. Let \circledast be a binary operation specified by $\mu \circledast \sigma = \sigma \circledast \mu, \forall \mu, \sigma \in [0,1]$. Consider fuzzy norm $\widetilde{\mathbb{N}}: \Upsilon \times (0, \infty) \to [0,1]$ represented in the following way:

 $\widetilde{N}(x, \tau) = \frac{\tau}{\tau + \|x\|}$ for all $x \in \Upsilon$ and $\tau > 0$ where $\|x\| = |x(t)|$

Let Γ , Λ : $\Upsilon \rightarrow \Upsilon$ specified by

$$\Gamma x(t) = \begin{cases} \frac{2x}{5} + \frac{1}{10} & \text{if } x \in [0,1] \\ \frac{3x}{4} + 3 & \text{if } x \in (1,\infty) \end{cases}$$

$$\Lambda y(t) = \begin{cases} \frac{2y}{5} + \frac{1}{10} & \text{if } y \in [0,1] \\ \frac{2y}{7} + \frac{60}{7} & \text{if } y \in (1,\infty) \end{cases}$$

Let $x, y \in \Upsilon$, we have

$$\frac{1}{\widetilde{N}(\Gamma_{X}(t) - \Lambda y(t), \tau)} - 1 = \frac{|\Gamma_{X}(t) - \Lambda y(t)|}{\tau}$$
$$= \frac{\left|\frac{2x}{5} + \frac{1}{10} - (\frac{2y}{5} + \frac{1}{10})\right|}{\tau}$$

$$= \frac{\left|\frac{2x}{5} - \frac{2y}{5}\right|}{\tau}$$

$$\leq \frac{2}{5} \frac{|x(s) - y(s)|}{\tau}$$

$$= \frac{2}{5} \left(\frac{1}{\tilde{N}(x(t) - y(t), \tau)} - 1\right)$$

Thus, the operators Γ and Λ meet the requirements of Theorem 1 with $\lambda_1 = \frac{2}{5} \in (0,1)$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$ in (1). Then Γ and Λ have a CF-point in Υ

Corollary 1: Assume that $(L, \widetilde{N}, \circledast)$ is a fuzzy Banach space and \widetilde{N} is triangular. Let $\Gamma, \Lambda: L \to L$ be a self-mapping with

$$\begin{split} \frac{1}{\widetilde{N}(\Gamma\mathbf{x} - \Lambda \boldsymbol{y}, \tau)} - 1 &\leq \lambda_1 \left(\frac{1}{\widetilde{N}(\mathbf{x} - \boldsymbol{y}, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(\mathbf{x} - \boldsymbol{y}, \tau)}{\widetilde{N}(\mathbf{x} - \Lambda \boldsymbol{y}, 2\tau) \circledast \widetilde{N}(\boldsymbol{y} - \Gamma \mathbf{x}, 2\tau)} - 1 \right) \\ &+ \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x} - \Gamma \mathbf{x}, \tau)} - 1 + \frac{1}{\widetilde{N}(\boldsymbol{y} - \Lambda \boldsymbol{y}, \tau)} - 1 \right) \end{split}$$

for all x, $\psi \in L$, $\tau > 0$, $\lambda_1 \in (0,1)$ and $\lambda_2, \lambda_4 \ge 0$ with $\lambda_1 + \lambda_2 + 2\lambda_4 < 1$. Then Γ and Λ possess a unique CF-point in L.

Proof: ψ_1 is a CF-point of Γ and Λ such that $\Gamma \psi_1 = \Lambda \psi_1 = \psi_1$ as shown by the Theorem 1 proof. For uniqueness, let x_1 be another CF-point of Γ and Λ in L such that $\Gamma x_1 = \Lambda x_1 = x_1$. Then,

$$\begin{split} \frac{1}{\widetilde{N}(\mathbf{x}_{1}-\boldsymbol{y}_{1},\tau)}-1 &= \frac{1}{\widetilde{N}(\Gamma\mathbf{x}_{1}-\boldsymbol{\Lambda}\boldsymbol{y}_{1},\tau)}-1 \\ &\leq \lambda_{1}\left(\frac{1}{\widetilde{N}(\mathbf{x}_{1}-\boldsymbol{y}_{1},\tau)}-1\right) + \lambda_{2}\left(\frac{\widetilde{N}(\mathbf{x}_{1}-\boldsymbol{y}_{1},\tau)}{\widetilde{N}(\mathbf{x}_{1}-\boldsymbol{\Lambda}\boldsymbol{y}_{1},2\tau)\circledast\widetilde{N}(\boldsymbol{y}_{1}-\Gamma\mathbf{x}_{1},2\tau)}-1\right) \\ &+ \lambda_{4}\left(\frac{1}{\widetilde{N}(\mathbf{x}_{1}-\Gamma\mathbf{x}_{1},\tau)}-1+\frac{1}{\widetilde{N}(\boldsymbol{y}_{1}-\boldsymbol{\Lambda}\boldsymbol{y}_{1},\tau)}-1\right) \\ Since \ \widetilde{N}(\boldsymbol{y}_{1}-\Gamma\mathbf{x}_{1},2\tau) \geq \widetilde{N}(\boldsymbol{y}_{1}-\mathbf{x}_{1},\tau)\circledast\widetilde{N}(\mathbf{x}_{1}-\Gamma\mathbf{x}_{1},\tau) \\ &= \widetilde{N}(\boldsymbol{y}_{1}-\mathbf{x}_{1},\tau)\circledast 1 \\ &= \widetilde{N}(\boldsymbol{y}_{1}-\mathbf{x}_{1},\tau) \end{split}$$

and

$$\begin{split} \widetilde{N}(\mathbf{x}_1 - \boldsymbol{\Lambda}\boldsymbol{y}_1, 2\tau) &\geq \widetilde{N}(\mathbf{x}_1 - \boldsymbol{y}_1, \tau) \circledast \widetilde{N}(\boldsymbol{y}_1 - \boldsymbol{\Lambda}\boldsymbol{y}_1, \tau) \\ &= \widetilde{N}(\mathbf{x}_1 - \boldsymbol{y}_1, \tau) \circledast \mathbf{1} \\ &= \widetilde{N}(\mathbf{x}_1 - \boldsymbol{y}_1, \tau) \end{split}$$

Consequently,

$$\frac{1}{\widetilde{N}(x_1 - \psi_1, \tau)} - 1 \le \lambda_1 \left(\frac{1}{\widetilde{N}(x_1 - \psi_1, \tau)} - 1 \right) + \lambda_2 \left(\frac{\widetilde{N}(x_1 - \psi_1, \tau)}{\widetilde{N}(x_1 - \psi_1, \tau) \circledast \widetilde{N}(\psi_1 - x_1, \tau)} - 1 \right)$$

$$\begin{split} & + \lambda_4 \left(\frac{1}{\widetilde{N}(\mathbf{x}_1 - \mathbf{x}_1, \tau)} - 1 + \frac{1}{\widetilde{N}(\mathbf{y}_1 - \mathbf{y}_1, \tau)} - 1 \right) \\ & = (\lambda_1 + \lambda_2) \left(\frac{1}{\widetilde{N}(\mathbf{x}_1 - \mathbf{y}_1, \tau)} - 1 \right) \\ & = (\lambda_1 + \lambda_2) \left(\frac{1}{\widetilde{N}(\Gamma \mathbf{x}_1 - \Lambda \mathbf{y}_1, \tau)} - 1 \right) \\ & \leq (\lambda_1 + \lambda_2)^2 \left(\frac{1}{\widetilde{N}(\mathbf{x}_1 - \mathbf{y}_1, \tau)} - 1 \right) \\ & \vdots \\ \end{split}$$

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$$\leq (\lambda_1 + \lambda_2)^j \left(\frac{1}{\widetilde{N}(\mathbf{x}_1 - \mathbf{y}_1, \tau)} - 1\right)$$

$$\rightarrow 0 \ as \ j \rightarrow \infty$$

$$since \ \lambda_1 + \lambda_2 < 1, \ therefore \ \widetilde{N}(\mathbf{x}_1 - \mathbf{y}_1, \tau) = 1. \ Hence \ \mathbf{x}_1 = \mathbf{y}_1 \ for \ \tau > 0.$$

Applications:

As an application of Theorem 1, we examine the existence and unique solution to Fredholm integral equations in this section. Consider the space $Y = C([0, a], \mathbb{R})$ which represents the space of all real-valued continuous functions on the interval [0, a], where $0 < a \in \mathbb{R}$.

The Fredholm integral equations are

$$\mathbf{x}(t) = \int_0^a f_1(t, s, \mathbf{x}(s)) ds \qquad \dots (8)$$

$$x(t) = \int_0^a f_2(t, s, x(s)) ds$$

where $t \in [0, a]$ and $f: [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$. The binary operation \circledast is specified by $\mu \circledast \sigma = \sigma \circledast \mu, \forall \mu, \sigma \in [0, a]$. The standard fuzzy norm $\widetilde{\mathbb{N}}: \Upsilon \times (0, \infty) \to [0, 1]$ represented in the following way:

$$\widetilde{N}(x,\tau) = \frac{\tau}{\tau + \aleph(x)}$$
 for all $x \in \Upsilon$ and $\tau > 0$ where $\aleph(x) = ||x|| = |x(t)|$.

Theorem 2: Suppose that

(i) there is a function h: $[0, a] \times [0, a] \to [0, \infty)$ which is continuous and $\lambda_1 \in (0,1)$ such that for all $x, y \in Y$ we have

$$\left| f_1(t, s, \mathbf{x}(s)) - f_2(t, s, y(s)) \right| \le \lambda_1 \, \mathbf{h}(t, s) |\mathbf{x}(s) - y(s)|$$
(ii) $\int_0^a \mathbf{h}(t, s) ds \le 1$

Then equation(8) possesses a unique solution in Υ .

Proof:

Let Γ , Λ : $\Upsilon \rightarrow \Upsilon$ specified by

$$\Gamma x(t) = \int_0^a f_1(t, s, x(s)) ds$$
$$\Lambda x(t) = \int_0^a f_2(t, s, x(s)) ds$$

Let $x, y \in \Upsilon$, we have

$$\begin{split} \frac{1}{\widetilde{\mathbf{N}}(\Gamma\mathbf{x}(t) - \Lambda \boldsymbol{y}(t), \tau)} - 1 &= \frac{|\Gamma\mathbf{x}(t) - \Lambda \boldsymbol{y}(t)|}{\tau} \\ &= \frac{\left| \int_0^a \mathbf{f}_1(t, s, \mathbf{x}(s)) ds - \int_0^a \mathbf{f}_2(t, s, \boldsymbol{y}(s)) ds \right|}{\tau} \\ &\leq \frac{\int_0^a \left| \mathbf{f}_1(t, s, \mathbf{x}(s)) - \mathbf{f}_2(t, s, \boldsymbol{y}(s)) \right| ds}{\tau} \\ &\leq \frac{\int_0^a \lambda_1 \mathbf{h}(t, s) |\mathbf{x}(s) - \boldsymbol{y}(s)| ds}{\tau} \\ &\leq \lambda_1 \frac{|\mathbf{x}(s) - \boldsymbol{y}(s)|}{\tau} \int_0^a \mathbf{h}(t, s) ds \end{split}$$

$$\leq \lambda_1 \frac{|x(s) - y(s)|}{\tau}$$
$$= \lambda_1 \left(\frac{1}{\widetilde{N}(x(t) - y(t), \tau)} - 1 \right)$$

Thus, the operators Γ meet the requirements of Theorem 1 with $\lambda_1 \in (0,1)$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$ in (1). Then Γ and Λ have a CF-point in Υ .

Conclusions

We introduce the triangle property on a fuzzy norm in this study and then use it to show the CF-point theorems in Fn-space. In addition to the above, we extended our results to the Fredholm equation to demonstrate the existence of an integral equation solution within the framework of Fn- spaces. It is a fascinating open problem to discover solutions to other kinds of equations on Fn-spaces.

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نظريات النقطة الصامدة المشتركة في الفضاء المعياري الضبابي

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معلومات المؤلف

الايميل:

الموبايل:

الخلاصة:

تعتبر نظرية النقطة الصامدة موضوعًا مثيرًا للاهتمام وله العديد من التطبيقات في العديد من فروع الرياضيات. بالإضافة إلى ذلك، توفر نظرية النقطة الصامدة طرقًا مفيدة لحل المشكلات والتي يمكن استخدامها عبر العديد من المجالات الفرعية للتحليل الرياضي. تهتم نظريات النقطة الصامدة بالدالة f للمجموعة X الى نفسها والتي، في ظل ظروف معينة، تسمح بنقطة صامدة ، أي نقطة XEX بحيث f(x)=x. في هذا البحث، نستخدم خاصية المثلث على الفضاء المعياري الضبابي (Fn-space) لإظهار نقطة صامدة مشتركة (CF-point) بدون استمرارية. أولاً، نراجع بعض المصطلحات الأساسية المستخدمة في السياق الضبابي. بعد ذلك يتم إعطاء فكرة الخاصية المثلثية للمعيار الضبابي ثم نثبت أن الدوال الذاتية Γ و Λ لها نقطة Γ في إعداد Fn-space. بالإضافة إلى ذلك، تم دراسة بعض تطبيقات نتائجنا الرئيسية على معادلة فريدهو لم التكاملية.