

Sum Graph of $Z_{p^n q^m}$ Groups and Some Topological Indices of G_+ ($Z_{p^n q^m}$)

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Abstract

This study introduces the extended graph theory in the sum group via $Z_{p^n q^m}$, where p and q are prime integers, and the sum is more than the order of the group $Z_{p^n q^m}$. If the group sum graph of $Z_{p^n q^m}$ it meets the properties of the graph theory and computes all the degrees of graphs, and it will be linked, cyclic, etc. will also use a generalized version of it to calculate the well-known topological indices.

Introduction:

Let G be a graph, and $G(V(G), E(G))$ be the set of its vertices and edges. This study solely takes into account straightforward finite graphs. $\text{Deg}(u)$ (or $d(u)$) of G is used to indicate a vertex $u \in V(G)$ degree. If an edge with the symbol $x \sim y$ connects two vertices, x and y , they are said to be adjacent (or $x \sim y$). The edges number connected to an edge refers to G as a graph having k vertices and l edges when saying that $G \in G + (k, l)$. The notations and terminologies used in this article but not defined in [1]. The numerical values connected to a graph structure are known as topological indices.

Numerous topological indices based on distance, degree, and other graph features have been suggested and investigated. You can find some of them in [2, 3]. Zagreb indices are historically regarded as the first degree-based topological indices, and they apply to many fields of science, including physical, chemical, and economic sciences etc. [4-7] Numerous works relate graph theory to group theory or ring theory features of the Z modulo; for further information, see [8-15] and the therein references. Alwardi A. et al. examined all of the graphs in the indices of Zagreb in 2018 [16].

The Sum graph of a group $G_+(Z_{p^n q^m})$ is defined in this study as a finite group of order $(Z_{p^n q^m})$ since p and q are prime integers numbers, and the vertices of the graph represent the

members of $(Z_{p^n q^m})$, such that an edge exists between the two vertices a and b if and only if $O(a)+O(b)>O(Z_{p^n q^m})$ denoted by $G_+(Z_{p^n q^m})$. The topological indices have received much study; some of these studies can be found in [17–20]. By Ranjini et al. in 2013, the index of Zagreb re defined as the first, second, and third indices of Zagreb [4]. The generalized inverse sum index ISI of graph G introduced by Buragohain et al. in [5] can also be considered a particular case of this index.

The Eccentric Connectivity Index, First and Second indices of Zagreb, Sum-Connectivity index, Randić' index (or Connectivity index), and sum particular special instances indices of $G_+(Z_{p^n q^m})$ are some of the topological indices computed in this study. Also obtained some novel features of $G_+(Z_{p^n q^m})$.

2. Basic Concepts and Terminology

The next main topic is the topological index that will be required in the following considerations.

The eccentric connectivity index \mathfrak{S}^c is defined as

$$\mathfrak{S}^c = \mathfrak{S}^c(G) = \sum_{i=1}^k e(u)d_i, \text{ where } e(u) = \max\{d(u,v), v \in V(G)\}. \quad [21]$$

generalized the Eccentric Connectivity index \mathfrak{S}_γ^c Is define as

$$\mathfrak{S}_\gamma^c = \mathfrak{S}_\gamma^c(G) = \sum_{i=1}^k e(u)d_i^\gamma, \gamma \in \mathbb{R}. \quad [21]$$

The first general Zagreb index (or general Zeroth-order Randić index) Q_γ is defined as

$$Q_\gamma = Q_\gamma(G) = \sum_{i=1}^k d_i^\gamma = \sum_{i \sim j} (d_i^{\gamma-1} + d_j^{\gamma-1}),$$

Where k is the order of vertices and $\gamma \in \mathbb{R}$ [22].

The second general Zagreb index or Randić index R_γ , is defined as

$$R_\gamma = R_\gamma(G) = \sum_{i \sim j} (d_i d_j)^\gamma,$$

And $\gamma \in \mathbb{R}$ [22].

The general Sum-Connectivity index H_μ as

$$H_\mu = H_\mu(G) = \sum_{i \sim j} (d_i + d_j)^\mu,$$

And $\mu \in \mathbb{R}$ [23].

3. The Sum graphs of the group $Z_{p^n q^m}$.

Shall explore the idea of a Z_n Group using graph theory in the following section by defining it in terms of order rules. Additionally, discovered the degree of the Z_n exceptional value at $(p^n q^m)$, where p and q are the prime numbers raised to the power of the positive integers n, m .

Definition 3.1 [24]: Let G be a finite cyclic group. The group sum graph, denoted by $G_+(V, E)$ of G_+ Is a graph with $V(G_+) = \cup_{x \in G} \langle x \rangle$ and two distinct vertices x and y are adjacent in G_+ , denoted

by $\langle x \rangle \sim \langle y \rangle$ if and only if $O(x) + O(y) > O(G)$, where $O(G)$ is the order of the group G . (i.e.)

$$V(G_+) = \cup_{x \in G} \langle x \rangle,$$

$$E(G_+) = \{xy | \langle x \rangle \sim \langle y \rangle \text{ if and only if } O(x) + O(y) > O(G), \text{ where } x, y \in G \text{ and } x \neq y\}.$$

Remark 3.2 [24]:

If taking as (definition 3.1) $O(x) + O(y) \leq O(G)$, where G is a finite group of order n . Because there is at least one member, we can observe that the graph is not connected. a such that $O(a) = O(G)$. Therefore, $O(a) + O(a_i) > O(G), \forall a_i \in G, 1 \leq i \leq n$, (i.e.) a is isolated-vertex. Hence G is not connected.

Lemma 3.3: Every finite cyclic group holds the group sum graph are connected and cyclic graphs.

Lemma 3.4 [24]: Let Z_p be a finite group of order a prime number p , then

$$G_+(V(Z_p), E(Z_p)) \cong K_p,$$

where K_p it is a complete graph of p vertices.

Theorem 3.5 [24]: Let $G_+(V(Z_{p^n}), E(Z_{p^n}))$, then

$$\deg(u) = \begin{cases} p^{n-1}(p-1) & \text{if } \mathcal{O}(u) \neq p^n \\ p^n - 1 & \text{if } \mathcal{O}(u) = p^n \end{cases},$$

Where $p \geq 3$ is the prime number and $n \geq 2$ is a positive integer number.

Proposition 3.6 [25]: If $G_+(V(Z_{pq}), E(Z_{pq}))$, then

$$\deg(u) = \begin{cases} pq - (p+q) + 1 & \text{if } \mathcal{O}(u) \neq pq \\ pq - 1 & \text{if } \mathcal{O}(u) = pq \end{cases},$$

where $u \in V(Z_{pq})$ And $p, q \geq 3$, p , and q are distinct prime numbers.

Theorem 3.7 [25]: Let $G_+(V(Z_{p^nq}), E(Z_{p^nq}))$, then

$$\deg(u) = \begin{cases} p^{n-1}(pq - (p+q) + 1) & \text{if } \mathcal{O}(u) \neq p^nq \\ p^nq - 1 & \text{if } \mathcal{O}(u) = p^nq \end{cases},$$

where $2 < p < q$ are distinct relative prime numbers, $n \geq 1, n \in \mathbb{Z}^+$.

Theorem 3.8[26]: Let $G_+(V(Z_{pq^m}), E(Z_{pq^m}))$, then

$$\deg(u) = \begin{cases} q^{m-1}(pq - (p+q) + 1) & \text{if } \mathcal{O}(u) \neq pq^m \\ pq^m - 1 & \text{if } \mathcal{O}(u) = pq^m \end{cases},$$

where, $2 < p < q, m \geq 1, m \in \mathbb{Z}^+$

Remark 3.9: We can generalize the cardinal number of $Z_{p^nq^m}$, by the following:

$$u_{u \in V(Z_{p^nq^m})}(u) = \begin{cases} 1 + \sum_{i=1}^{\Delta} \alpha_i = p^{n-1}q^{m-1}(p+q-1) & \text{if } \mathcal{O}(u) \neq p^nq^m \\ p^nq^m - [1 + \sum_{i=1}^{\Delta} \alpha_i] = p^{n-1}q^{m-1}(pq - (p+q) + 1) & \text{if } \mathcal{O}(u) = p^nq^m \end{cases},$$

where $\Delta = (nm + n + m - 1), n, m \geq 1, n, m \in \mathbb{Z}^+$ And $2 < p < q$ different prime numbers.

Theorem 3.10: Let $G_+(V(Z_{p^nq^m}), E(Z_{p^nq^m}))$, then

$$\deg(u) = \begin{cases} p^{n-1}q^{m-1}(pq - (p+q) + 1) & \text{if } \mathcal{O}(u) \neq p^nq^m \\ p^nq^m - 1 & \text{if } \mathcal{O}(u) = p^nq^m \end{cases},$$

$2 < p < q$ are distinct prime numbers and $n, m \geq 1$ are positive integer numbers.

Proof: The proof follows immediately from Proposition 3.6, Theorems 3.5, 3.7, and 3.8 have the result.

Remark 3.11:

$$\sum_{u \in V(Z_{p^nq^m})} \deg(u) = [p^{n-1}q^{m-1}(pq - (p+q) + 1)][p^{n-1}q^{m-1}(pq + p + q - 1) - 1].$$

$n, m \geq 1$.

$$l = \frac{1}{2} [p^{n-1}q^{m-1}(pq - (p + q) + 1)][p^{n-1}q^{m-1}(pq + p + q - 1) - 1].$$

Remark 3.12:

1-The Sum graph $G_+(Z_{p^nq^m})$ is Hamilton's graph because the degree of any vertex is greater or equal to $p^nq^m/2$.

2- The Sum graph $G_+(Z_{p^nq^m})$ is an Euler's graph because all vertices have an even degree.

Remark 3.13: We can express the graph $G_+(V(Z_{p^nq^m}), E(Z_{p^nq^m}))$ by the formal

$K_{p^{n-1}q^{m-1}(pq-(p+q)+1)} + S_{p^{n-1}q^{m-1}(p+q-1)}$, where any $v \in V(K_{p^{n-1}q^{m-1}(pq-(p+q)+1)})$ have $\mathcal{O}(v) = p^nq^m$ and $u \in V(S_{p^{n-1}q^{m-1}(p+q-1)})$ have $\mathcal{O}(u) \neq p^nq^m$, $3 \leq p < q$ are prime numbers and $n, m \geq 1$ are positive integer number.

4. Topological indices of $G_+(Z_{p^nq^m})$

This section will compute the famous generalized topological indices with exceptional cases.

Remarks 4.1:

(1) Let $u \in V(G)$, where G_+ is a finite simple graph of order $(n \neq p)$, where p is a prime number.

$$e(u) = \begin{cases} 1 & \text{if } \mathcal{O}(u) = |Z_n| \\ 2 & \text{if } \mathcal{O}(u) \neq |Z_n| \end{cases}$$

(2) The Eccentric Connectivity index \mathfrak{S}^c Is define as

$$\mathfrak{S}^c = \mathfrak{S}^c(G) = \sum_{i=1}^k e(u)d_i.$$

(3) The general Eccentric Connectivity index \mathfrak{S}_γ^c Is define as

$$\mathfrak{S}_\gamma^c = \mathfrak{S}_\gamma^c(G) = \sum_{i=1}^k e(u)d_i^\gamma, \gamma \in R.$$

Theorem 4.2.: The eccentric connectivity index of the graph $G_+(V(Z_{p^nq^m}), E(Z_{p^nq^m}))$ is

$$\begin{aligned} \mathfrak{S}^c(Z_{p^nq^m}) &= \sum_{i=1}^{p^nq^m} e(u)d_i \\ &= p^{n-1}q^{m-1}(pq - (p + q) + 1)[2p^{n-1}q^{m-1}(p + q - 1) + (p^nq^m - 1)]. \end{aligned}$$

Proof: from remark 3.9, the vertices of $G_+(V(Z_{p^nq^m}), E(Z_{p^nq^m}))$ have a degree by theorem 3.10 where $2 < p < q$, and n, m are positive integers numbers, $n, m \geq 1$, get

$$\begin{aligned} \mathfrak{S}^c(Z_{p^nq^m}) &= 2 \left[\frac{(p^{n-1}q^{m-1}(pq - (p + q) + 1)) + \dots + (p^{n-1}q^{m-1}(pq - (p + q) + 1))}{p^{n-1}q^{m-1}(p+q-1)\text{-times}} \right] \\ &+ 1 \left[\frac{(p^nq^m - 1) + (p^nq^m - 1) + \dots + (p^nq^m - 1)}{p^{n-1}q^{m-1}(pq-(p+q)+1)\text{-times}} \right] \\ &= 2p^{n-1}q^{m-1}(p + q - 1)(p^{n-1}q^{m-1}(pq - (p + q) + 1)) + p^{n-1}q^{m-1}(pq - (p + q) + 1)(p^nq^m - 1) \\ \mathfrak{S}^c(Z_{p^nq^m}) &= p^{n-1}q^{m-1}(pq - (p + q) + 1)[2p^{n-1}q^{m-1}(p + q - 1) + (p^nq^m - 1)]. \end{aligned}$$

Theorem:4.3: The general Eccentric connectivity index of the graph $G_+(V(Z_{p^nq^m}), E(Z_{p^nq^m}))$ is

$$\mathfrak{S}_\gamma^c(Z_{p^nq^m}) = \sum_{i=1}^{p^nq^m} e(u)d_i^\gamma = p^{n-1}q^{m-1}(pq - (p + q) + 1)[2p^{n-1}q^{m-1}(p + q - 1)(p^{n-1}q^{m-1}(pq - (p + q) + 1))^{\gamma-1} + (p^nq^m - 1)^\gamma], \gamma \in R.$$

Proof: From remark 3.9 and by theorem 3.10, get

$$\mathfrak{S}_\gamma^c(Z_{p^n q^m}) = 2 \left[\underbrace{\left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^\gamma + \dots + \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^\gamma}_{p^{n-1} q^{m-1} (p+q-1)\text{-times}} \right] \\ + 1 \left[\underbrace{\left(p^n q^m - 1 \right)^\gamma + \left(p^n q^m - 1 \right)^\gamma + \dots + \left(p^n q^m - 1 \right)^\gamma}_{p^{n-1} q^{m-1} (pq - (p+q) + 1)\text{-times}} \right]$$

$$= 2p^{n-1} q^{m-1} (p+q-1) \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^\gamma + p^{n-1} q^{m-1} (pq - (p+q) + 1) (p^n q^m - 1)^\gamma.$$

$$\mathfrak{S}_\gamma^c(Z_{p^n q^m}) = p^{n-1} q^{m-1} (pq - (p+q) + 1) \left[2p^{n-1} q^{m-1} (p+q-1) \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^{\gamma-1} + (p^n q^m - 1)^\gamma \right].$$

Example 4.4.

$$Z_3^3 \stackrel{3}{=} \{0, 1, 2, \dots, 3374\}, p=3, q=5, m=3, n=3, V(G)=3375.$$

$$\mathfrak{S}^c(Z_{p^n q^m}) = 11743200.$$

Theorem 4.5.: Let $G_+(V(Z_{p^n q^m}), E(Z_{p^n q^m}))$, then the first Zagreb index is

$$M_1 = M_1(Z_{p^n q^m}) = p^{n-1} q^{m-1} (pq - (p+q) + 1) \left[p^{n-1} q^{m-1} (p+q-1) \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right) + (p^n q^m - 1)^2 \right].$$

Proof: The vertices of $G_+(V(Z_{p^n q^m}), E(Z_{p^n q^m}))$ have a degree by theorem 3.10 where $2 < p < q$, and n, m are positive integers numbers, $n, m \geq 1$, get

$$M_1 = M_1(Z_{p^n q^m}) = \sum_{i=1}^{p^n q^m} d_i^2 = \\ = \underbrace{\left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^2 + \dots + \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^2}_{p^{n-1} q^{m-1} (p+q-1)\text{-times}} \\ + \underbrace{\left(p^n q^m - 1 \right)^2 + \left(p^n q^m - 1 \right)^2 + \dots + \left(p^n q^m - 1 \right)^2}_{p^{n-1} q^{m-1} (pq - (p+q) + 1)\text{-times}}. \\ = p^{n-1} q^{m-1} (p+q-1) \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^2 + p^{n-1} q^{m-1} (pq - (p+q) + 1) (p^n q^m - 1)^2.$$

$$M_1(Z_{p^n q^m}) = p^{n-1} q^{m-1} (pq - (p+q) + 1) \left[p^{n-1} q^{m-1} (p+q-1) \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right) + (p^n q^m - 1)^2 \right].$$

Theorem 4.6.: Let $G_+(V(Z_{p^n q^m}), E(Z_{p^n q^m}))$, then the first general Zagreb index is

$$Q_\gamma = p^{n-1} q^{m-1} (pq - (p+q) + 1) \left[p^{n-1} q^{m-1} (p+q-1) \left(p^{n-1} q^{m-1} (pq - (p+q) + 1) \right)^{\gamma-1} + (p^n q^m - 1)^\gamma \right].$$

Proof: The vertices of $G_+(V(Z_{p^n q^m}), E(Z_{p^n q^m}))$ have a degree by theorem 3.10 where $2 < p < q$, and n, m are positive integers numbers, $n, m \geq 1, \gamma \in R$ get

$$Q_\gamma = Q_\gamma(Z_{p^n q^m}) = \sum_{i=1}^{p^n q^m} d_i^\gamma =$$

$$= \underbrace{\left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right)^\gamma + \dots + \left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right)^\gamma}_{p^{n-1}q^{m-1}(p+q-1)\text{-times}}$$

$$+ \underbrace{\left(p^nq^m - 1 \right)^\gamma + \left(p^nq^m - 1 \right)^\gamma + \dots + \left(p^nq^m - 1 \right)^\gamma}_{p^{n-1}q^{m-1}(pq-(p+q)+1)\text{-times}}.$$

$$= p^{n-1}q^{m-1}(p+q-1)\left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right)^\gamma + p^{n-1}q^{m-1}(pq-(p+q)+1)(p^nq^m-1)^\gamma.$$

$$Q_\gamma(Z_{p^nq^m}) = p^{n-1}q^{m-1}(pq-(p+q)+1) \left[p^{n-1}q^{m-1}(p+q-1)\left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right)^{\gamma-1} + (p^nq^m-1)^\gamma \right].$$

In particular

$$(1) \text{ If } \gamma = 1 \Rightarrow Q_1 = \sum_{i=1}^{p^nq^m} d_i = 2\ell (\ell \text{ is a size of graph})$$

$$\Rightarrow \ell = \frac{1}{2}Q_1 = \frac{1}{2} \left[p^{n-1}q^{m-1}(pq-(p+q)+1) \left[p^{n-1}q^{m-1}(p+q-1) + (p^nq^m-1) \right] \right]$$

$$(2) \text{ If } \gamma = 2 \Rightarrow Q_2 = \sum_{i=1}^{p^nq^m} d_i^2 = \sum_{i \sim j} (d_i + d_j) = M_1$$

$$= p^{n-1}q^{m-1}(pq-(p+q)+1) \left[p^{n-1}q^{m-1}(p+q-1) \cdot \left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right) + (p^nq^m-1)^2 \right]$$

$$(3) \text{ If } \gamma = 3 \Rightarrow Q_3 = \sum_{i=1}^{p^nq^m} d_i^3 = \sum_{i \sim j} (d_i^2 + d_j^2) = F_1 (\text{forging index})$$

$$= p^{n-1}q^{m-1}(pq-(p+q)+1) \left[p^{n-1}q^{m-1}(p+q-1) \cdot \left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right)^2 + (p^nq^m-1)^3 \right].$$

Example 4.7.

$$Z_3^3 = \{0, 1, 2, \dots, 3374\}, p=3, q=5, m=3, n=3, V(G)=3375$$

$$(1) \text{ If } \gamma = 1 \Rightarrow Q_1 = \sum_{i=1}^{p^nq^m} d_i = 2\ell (\ell \text{ is a size of graph})$$

$$\Rightarrow \ell = \frac{1}{2}Q_1 = 4454100.$$

$$(2) \text{ If } \gamma = 2 \Rightarrow Q_2 = \sum_{i=1}^{p^nq^m} d_i^2 = \sum_{i \sim j} d_i + d_j = M_1$$

$$= p^{n-1}q^{m-1}(pq-(p+q)+1) \left[p^{n-1}q^{m-1}(p+q-1) \cdot \left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right) + (p^nq^m-1)^2 \right] = 25593976800.$$

$$(3) \text{ If } \gamma = 3 \Rightarrow Q_3 = \sum_{i=1}^{p^nq^m} d_i^3 = \sum_{i \sim j} (d_i^2 + d_j^2) = F$$

$$= p^{n-1}q^{m-1}(pq-(p+q)+1) \left[p^{n-1}q^{m-1}(p+q-1) \cdot \left(p^{n-1}q^{m-1}(pq-(p+q)+1) \right)^2 + (p^nq^m-1)^3 \right] = 87507355723200.$$

Remark 4.8:

(1) The second Zagreb's index is defined as

$$M_2 = M_2(G) = \sum_{i \sim j} (d_i d_j)$$

(2) The generalized second Zagreb's index or Randic index R_γ , defined as

$$R_\gamma = R_\gamma(G) = \sum_{i \sim j} (d_i d_j)^\gamma, \gamma \in \mathbb{R}$$

Theorem 4.9.: Let $G_+ (V(Z_{p^nq^m}), E(Z_{p^nq^m}))$, then

$$M_2(Z_{p^nq^m}) = R_1 = \sum_{i \sim j} d_i d_j = p^{n-1}q^{m-1}(pq-(p+q)+1)(p^nq^m-1) \cdot \left[\left[p^{n-1}q^{m-1}(pq-(p+q)+1) \right] \left[(p^{n-1}q^{m-1}(pq-(p+q)+1) - 1) \right] - (p^nq^m-1) \frac{(p^{n-1}q^{m-1}(pq-(p+q)+1)+1)}{2} \right]$$

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}$, $\alpha = p^{n-1}q^{m-1}(pq - (p + q) + 1)$,
 $S \subseteq Z_{p^n q^m}$, where $a_i \in Z_{p^n q^m}$, $\mathcal{O}(a_i) = p^n q^m$, $1 \leq i \leq \alpha$.

see that, every $a_i \in S$, $1 \leq i \leq k$ is adjacent to all vertices belonging to $Z_{p^n q^m}$ it except itself.
(Since the graph is a simple graph).

$$R_1 = d(a_1) \left[\sum_{i=1}^{p^n q^m} d(i) - d(a_1) \right] + d(a_2) \left[\sum_{i=1}^{p^n q^m} d(i) - (d(a_1) + d(a_2)) \right] + \dots^{\alpha\text{-times}}$$

$$+ d(a_\alpha) \left[\sum_{i=1}^{p^n q^m} d_i - \sum_{i=1}^{\alpha} d(a_i) \right].$$

Now, since $d(a_1) = d(a_2) = \dots = d(a_\alpha) = d(a) = p^n q^m - 1$ and
 $\sum_{i=1}^{p^n q^m} d_i = 2\ell$ (ℓ is the size of a graph).

$$\Rightarrow R_1 = d(a) \left[\underbrace{(2\ell - d(a)) + (2\ell - 2d(a)) + \dots + (2\ell - \alpha d(a))}_{\alpha\text{-times}} \right] \dots (1)$$

$$= d(a) [\alpha \cdot 2\ell - d(a) \sum_{i=1}^{\alpha} i].$$

$$= d(a) \left[2\alpha\ell - d(a) \frac{\alpha(\alpha-1)}{2} \right], \quad \sum_{i=1}^{\alpha} i = \frac{\alpha(\alpha-1)}{2}.$$

$$= \alpha d(a) \left[2\ell - \frac{\alpha+1}{2} d(a) \right].$$

$$= p^{n-1}q^{m-1}(pq - (p + q) + 1)(p^n q^m - 1) \cdot \left[[p^{n-1}q^{m-1}(pq - (p + q) + 1)] [p^{n-1}q^{m-1}(pq + p + q - 1) - 1] - (p^n q^m - 1) \frac{p^{n-1}q^{m-1}(pq - (p + q) + 1) + 1}{2} \right].$$

So, in general, get by Eq. (1)

$$R_Y = (d(a))^Y \left[\underbrace{(2\ell - d(a))^Y + (2\ell - 2d(a))^Y + \dots + (2\ell - \alpha d(a))^Y}_{\alpha\text{-times}} \right].$$

$$R_Y = (d(a))^Y \sum_{i=1}^{\alpha} (2\ell - id(a))^Y$$

$$= (p^n q^m - 1)^Y \sum_{i=1}^{\alpha} \left[(p^{n-1}q^{m-1}(pq - (p + q) + 1)) (p^{n-1}q^{m-1}(pq + p + q - 1) - 1) - i(p^n q^m - 1) \right]^Y$$

In particular:

$$(1) R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j} = \frac{1}{(p^n q^m - 1)}$$

$$\sum_{i=1}^{p^{n-1}q^{m-1}(pq - (p + q) + 1)} \left[\frac{1}{p^{n-1}q^{m-1}(pq - (p + q) + 1)(p^{n-1}q^{m-1}(pq + p + q - 1) - 1) - i(p^n q^m - 1)} \right].$$

$$(2) R_{-1/2} = \lambda(Z_{p^n q^m}) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}} = \frac{1}{\sqrt{p^n q^m - 1}}$$

$$\sum_{i=1}^{p^{n-1}q^{m-1}(pq - (p + q) + 1)} \left[\frac{1}{\sqrt{p^{n-1}q^{m-1}(pq - (p + q) + 1)(p^{n-1}q^{m-1}(pq + p + q - 1) - 1) - i(p^n q^m - 1)}} \right].$$

Example 4.10.

$$Z_3^3 \frac{3}{5} = \{0, 1, 2, \dots, 3374\}, \quad p=3, q=5, m=3, n=3, V(G)=3375$$

$$(1) R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j} = \frac{1}{(3374)} \sum_{i=1}^{1800} \left[\frac{1}{1800 * (1799 - i(3374))} \right]$$

$$(2) R_{-1/2} = \lambda(Z_{p^n q^m}) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}} = \frac{1}{\sqrt{3374}} \sum_{i=1}^{1800} \left[\frac{1}{\sqrt{(1800)(1799 - i(3374))}} \right]$$

$$(3) R_1 = M_2(Z_{p^n q^m}) = \sum_{i \sim j} d_i d_j = 1,214,111,631,600$$

Remark 4.11:

(1) The Sum-Connectivity index χ defined as

$$\chi(G) = \sum_{i \sim j} (d_i + d_j)^{-\frac{1}{2}}$$

(2) the general Sum-Connectivity index k_μ defined as

$$k_\mu = k_\mu(G) = \sum_{i \sim j} (d_i + d_j)^\mu, \mu \in \mathbb{R}$$

Theorem 4.12: Let $G_+ (V(Z_{p^n q^m}), E(Z_{p^n q^m}))$, then

$$\begin{aligned} \chi(Z_{p^n q^m}) &= \sum_{i \sim j} (d_i + d_j)^{-\frac{1}{2}} \\ &= p^{n-1} q^{m-1} (pq - (p + q) + 1) \left[2^{-\frac{3}{2}} (p^{n-1} q^{m-1} (pq - (p + q) + 1) - 1) (p^n q^m - 1)^{-\frac{1}{2}} + \right. \\ &\quad \left. p^{n-1} q^{m-1} (p + q - 1) (p^n q^m - 1 + p^{n-1} q^{m-1} (pq - (p + q) + 1))^{-\frac{1}{2}} \right]. \end{aligned}$$

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}$, $\alpha = p^{n-1} q^{m-1} (pq - (p + q) + 1)$, $S \subseteq Z_{p^n q^m}$

$a_i \in Z_{p^n q^m}$, where $\mathcal{O}(a_i) = p^n q^m, \forall 1 \leq i \leq \alpha$. Since every a_i it is adjacent to all vertices to $Z_{p^n q^m}, \forall 1 \leq i \leq \alpha$ except itself, then:

$$\begin{aligned} \chi &= \underbrace{(d(a_1) + d(a_1))^{-\frac{1}{2}} + (d(a_1) + d(a_2))^{-\frac{1}{2}} + \dots + (d(a_1) + d(a_\alpha))^{-\frac{1}{2}}}_{\alpha\text{-times}} \\ &\quad - \underbrace{(d(a_1) + d(a_1))^{-\frac{1}{2}} + (d(a_1) + d(0))^{-\frac{1}{2}} + \dots + (d(a_1) + d(\alpha_\Delta p^{n-1} q^m))^{-\frac{1}{2}}}_{p^{n-1} q^{m-1} (p+q-1)\text{-times}} \\ &\quad + \underbrace{\ddots}_{\alpha\text{-times}} + \underbrace{(d(a_\alpha) + d(a_1))^{-\frac{1}{2}} + \dots + (d(a_\alpha) + d(a_\alpha))^{-\frac{1}{2}}}_{\alpha\text{-times}} \\ &\quad - \left[\underbrace{(d(a_\alpha) + d(a_1))^{-\frac{1}{2}} + (d(a_\alpha) + d(a_2))^{-\frac{1}{2}} + \dots + (d(a_\alpha) + d(a_\alpha))^{-\frac{1}{2}}}_{\alpha\text{-times}} \right] \\ &\quad + \left[\underbrace{(d(a_\alpha) + d(a_1))^{-\frac{1}{2}} + (d(a_\alpha) + d(p))^{-\frac{1}{2}} + \dots + (d(a_\alpha) + d(\alpha_\Delta p^{n-1} q^m))^{-\frac{1}{2}}}_{p^{n-1} q^{m-1} (p+q-1)\text{-times}} \right]. \end{aligned}$$

Where $\Delta = mn + n + m - 1$

now, since $d(a_1) = d(a_2) = \dots = d(a_\alpha) = d(a) = p^n q^m - 1$, and

$d(0) = d(p) = \dots = d(\alpha_\Delta p^{n-1} q^m) = p^{n-1} q^{m-1} (pq - (p + q) + 1)$.

$$\begin{aligned} \chi &= \left[(\alpha - 1) (2d(a))^{-\frac{1}{2}} + p^{n-1} q^{m-1} (p + q - 1) (d(a) + \alpha)^{-\frac{1}{2}} + [(\alpha - \right. \\ &\quad \left. 2) (2d(a))^{-\frac{1}{2}} p^{n-1} q^{m-1} (p + q - 1) (d(a) + \alpha)^{-\frac{1}{2}} \right] + \dots + \left[(\alpha - \alpha) (2d(a))^{-\frac{1}{2}} + p^{n-1} q^{m-1} (p + \right. \\ &\quad \left. q - 1) (d(a) + \alpha)^{-\frac{1}{2}} \right]. \end{aligned}$$

$$\chi = (2d(a))^{-\frac{1}{2}} \sum_{i=1}^{\alpha} (\alpha - i) + \alpha p^{n-1} q^{m-1} (p + q - 1) (d(a) + \alpha)^{-\frac{1}{2}}.$$

since $\sum_{i=1}^{\alpha} (\alpha - i) = \frac{\alpha(\alpha-1)}{2}$.

$$\chi = \alpha \left[\frac{(\alpha-1)}{2} (2d(a))^{-\frac{1}{2}} + p^{n-1} q^{m-1} (p + q - 1) (d(a) + \alpha)^{-\frac{1}{2}} \right].$$

$$= p^{n-1}q^{m-1}(pq - (p + q) + 1) \left[2^{-\frac{3}{2}}(p^{n-1}q^{m-1}(pq - (p + q) + 1) - 1)(p^nq^m - 1)^{-\frac{1}{2}} + p^{n-1}q^{m-1}(p + q - 1) \left(p^nq^m - 1 + p^{n-1}q^{m-1}(pq - (p + q) + 1) \right)^{-\frac{1}{2}} \right]$$

Theorem 4.13.: The general Sum-Connectivity Index of $G_+(V(Z_{p^nq^m}), E(Z_{p^nq^m}))$, defined as

$$k_\mu(Z_{p^nq^m}) = \sum_{i \sim j} (d_i + d_j)^\mu = p^{n-1}q^{m-1}(pq - (p + q) + 1) \left[2^{\mu-1}(p^{n-1}q^{m-1}(pq - (p + q) + 1) - 1)(p^nq^m - 1)^\mu + p^{n-1}q^{m-1}(p + q - 1) \left(p^nq^m - 1 + p^{n-1}q^{m-1}(pq - (p + q) + 1) \right)^\mu \right]$$

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}$, $\alpha = p^{n-1}q^{m-1}(pq - (p + q) + 1)$, $S \subseteq Z_{p^nq^m}$

$a_i \in Z_{p^nq^m}$, where $\mathcal{O}(a_i) = p^nq^m$, $\forall 1 \leq i \leq \alpha$. Since every a_i it is adjacent to all vertices to $Z_{p^nq^m}$, $\forall 1 \leq i \leq \alpha$ except itself, then:

$$\begin{aligned} k_\mu &= \underbrace{(d(a_1) + d(a_1))^\mu + (d(a_1) + d(a_2))^\mu + \dots + (d(a_1) + d(a_\alpha))^\mu}_{\alpha\text{-times}} \\ &\quad - \underbrace{(d(a_1) + d(a_1))^\mu + (d(a_1) + d(0))^\mu + \dots + (d(a_1) + d(\alpha_\Delta p^{n-1}q^m))^\mu}_{p^{n-1}q^{m-1}(p+q-1)\text{-times}} \\ &\quad + \underbrace{\ddots}_{\alpha\text{-times}} + \underbrace{(d(a_\alpha) + d(a_1))^\mu + \dots + (d(a_\alpha) + d(a_\alpha))^\mu}_{\alpha\text{-times}} \\ &\quad - \left[\underbrace{(d(a_\alpha) + d(a_1))^\mu + (d(a_\alpha) + d(a_2))^\mu + \dots + (d(a_\alpha) + d(a_\alpha))^\mu}_{\alpha\text{-times}} \right] \\ &\quad + \left[\underbrace{(d(a_\alpha) + d(a_1))^\mu + (d(a_\alpha) + d(p))^\mu + \dots + (d(a_\alpha) + d(\alpha_\Delta p^{n-1}q^m))^\mu}_{p^{n-1}q^{m-1}(p+q-1)\text{-times}} \right]. \end{aligned}$$

Where $\Delta = mn + n + m - 1$

Now, since $d(a_1) = d(a_2) = \dots = d(a_\alpha) = d(a) = p^nq^m - 1$, and $d(0) = d(p) = \dots = d(\alpha_\Delta p^{n-1}q^m) = p^{n-1}q^{m-1}(pq - (p + q) + 1)$.

$$k_\mu = \left[(\alpha - 1)(2d(a))^\mu + p^{n-1}q^{m-1}(p + q - 1)(d(a) + \alpha)^{-\frac{1}{2}} + [(\alpha - 2)(2d(a))^\mu p^{n-1}q^{m-1}(p + q - 1)(d(a) + \alpha)^\mu] + \dots + [(\alpha - \alpha)(2d(a))^\mu + p^{n-1}q^{m-1}(p + q - 1)(d(a) + \alpha)^\mu] \right].$$

$$k_\mu = (2d(a))^\mu \sum_{i=1}^{\alpha} (\alpha - i) + \alpha p^{n-1}q^{m-1}(p + q - 1)(d(a) + \alpha)^\mu.$$

$$\text{since } \sum_{i=1}^{\alpha} (\alpha - i) = \frac{\alpha(\alpha-1)}{2}.$$

$$k_\mu = \alpha \left[\frac{(\alpha - 1)}{2} (2d(a))^\mu + p^{n-1}q^{m-1}(p + q - 1)(d(a) + \alpha)^\mu \right].$$

$$= p^{n-1}q^{m-1}(pq - (p + q) + 1) \left[2^{\mu-1}(p^{n-1}q^{m-1}(pq - (p + q) + 1) - 1)(p^nq^m - 1)^\mu + p^{n-1}q^{m-1}(p + q - 1) \left(p^nq^m - 1 + p^{n-1}q^{m-1}(pq - (p + q) + 1) \right)^\mu \right].$$

In particular:

$$(1) \text{ If } \mu = 1 \Rightarrow k_1 = M_1 = Q_2 =$$

$$= p^{n-1}q^{m-1}(pq - (p + q) + 1)[(p^{n-1}q^{m-1}(pq - (p + q) + 1) - 1)(p^nq^m - 1) + p^{n-1}q^{m-1}(p + q - 1)(p^nq^m - 1 + p^{n-1}q^{m-1}(pq - (p + q) + 1))].$$

$$(2) \text{ If } \mu = -1 \Rightarrow 2k_{-1} = \sum_{i \sim j} \frac{2}{d_i + d_j} = 2 \left[p^{n-1}q^{m-1}(pq - (p + q) + 1) \cdot$$

$$\left[\frac{(p^{n-1}q^{m-1}(pq - (p + q) + 1) - 1)}{4(p^nq^m - 1)} + \frac{p^{n-1}q^{m-1}(p + q - 1)}{(p^nq^m - 1 + p^{n-1}q^{m-1}(pq - (p + q) + 1))} \right].$$

$$(3) \text{ If } \mu = -1/2 \Rightarrow k_{-1/2} = \chi(Z_{p^nq^m}) = \sum_{i \sim j} \frac{1}{\sqrt{d_i + d_j}}.$$

$$= p^{n-1}q^{m-1}(pq - (p + q) + 1) \cdot \left[\frac{(p^{n-1}q^{m-1}(pq - (p + q) + 1) - 1)}{(\sqrt{2})^3 \sqrt{p^nq^m - 1}} + \frac{p^{n-1}q^{m-1}(p + q - 1)}{\sqrt{p^nq^m - 1 + p^{n-1}q^{m-1}(pq - (p + q) + 1)}} \right].$$

Example 4.14.

$$Z_3^3 \times_5^3 = \{0, 1, 2, \dots, 3374\}, p=3, q=5, m=3, n=3, O(G)=3375$$

$$(1) \text{ If } \mu = 1 \Rightarrow k_1 = M_1 = Q_2 = 25593976800.$$

$$(2) \text{ If } \mu = -1 \Rightarrow 2k_{-1} = \sum_{i \sim j} \frac{2}{d_i + d_j} = 1575.739.$$

$$(3) \text{ If } \mu = -1/2 \Rightarrow k_{-1/2} = \chi(Z_{p^nq^m}) = \sum_{i \sim j} \frac{1}{\sqrt{d_i + d_j}} = 59123.01.$$

5. Conclusions

The analysis reveals a few explicit characteristics of the group $(Z_{p^nq^m})$ graphs, including the Eccentric Connectivity Index, the First and Second Zagreb indices, the Sum-Connectivity Index, the Randic Index (or Connectivity Index), and the Sum Particular Special Cases Indices of the group $(Z_{p^nq^m})$.

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بيان الجمع للزمرة المنتهية $Z_-(p^n q^m)$ وبعض المؤشرات الطوبولوجية لـ $G + (Z_-(p^n q^m))$

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البحث مستل من أطروحة الدكتوراه الباحث الاول

الخلاصة:

في هذه الدراسة، قدمنا نظرية البيان الموسعة في زمرة المجموع عبر $Z_-(p^n q^m)$ ، حيث p و q هما عدنان صحيحان أوليان، ويكون المجموع أكبر من ترتيب المجموعة $Z_-(p^n q^m)$. إذا كان بيان الجمع للزمرة $Z_-(p^n q^m)$ يفي بخصائص نظرية البيان ويحسب جميع درجات الرسوم البيانية، وسيكون مرتبطاً، دورياً، إلخ. سنستخدم أيضاً نسخة عامة منه لـ حساب المؤشرات الطوبولوجية المعروفة.

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