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Fundamental Theorems and Some Properties of Maximal $iD_g^{\#}$-Open Sets in quad topological spaces

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Abstract

This study deals with the fundamental properties of maximal $iD_g^{\#}$-open sets in field of topology. The focus is on a set of important theoretical results, including the proof of the decomposition theorem, which explains how these sets can be divided into simpler subsets while preserving their intrinsic properties. In addition, the study investigates the properties associated with the intersection of maximal $iD_g^{\#}$-open sets, with an emphasis on relationship between these intersections and the low of radical closure. Exact proofs of these properties are provided and their theoretical importance in understanding the dynamics of open sets in this type is demonstrated. The outcomes contribute to the mathematical understanding of topology by providing new analysis methods that can be applied to the study of a new class of open sets.

Introduction

In the field of topology, the study of open sets is the cornerstone for understanding the mathematical structures of spaces. These sets are based on a set of properties and rules that form the basis for analysing the relationships between points and sets in topological spaces. Among the important concepts in this field are the concepts of open sets and maximum open sets, which represent new challenges in the analysis of general topology and its relationship to spaces with complex structure.

Therefore, many researches were presented in the field of new open sets, including: a new set of regular general open sets was presented and how they interact with other open and semi-open sets using examples and proofs [1], the concept of semi-open sets in soft topological spaces was developed, with its applications in dealing with uncertainty in many fields [2], a new definition of C-Open Sets was presented, and their properties related to separation, compactness, and continuity were explored. It also presents some important theoretical results using illustrative examples [3], and the concepts of separation and continuity are studied using semi-primary sets [4], Soft ωδ-Open Sets [5], and it also highlights a new type of semi-open sets (Nearly Open Sets) and explains their properties [6], and the analysis of general open sets using partition theories that provide a deeper insight into the relationships of these sets in different topological spaces [7].

This study aims to find some properties of $iD_g^{\#}$-open sets, the $iD_g^{\#}$-open sets and $iD_g^{\#}$-minimal open sets and use them to get some findings $iD_g^{\#}$-open sets and pre-Hausdorff spaces.

Section 2 introduces some fundamental Concepts. In section 3, study and focused on maximal $iD_g^{\#}$. In section 4, study Closure, interior, and ${iD_g^{\#}}_{Max}$-open sets, while in section 4, study Fundamental properties of radicals, finally in section 5 focused on additional on radicals of ${iD_g^{\#}}_{Max}$-open sets.

1. Fundamental Concepts

Definition (1.1) [8],[9] : A non-empty set X containing four universal topologies denoted by T_1, T_2, T_3 and T_4 then a subset A of X is called quad-open (φ-open) if the condition is holds: A subset of $T_1 \cup T_2 \cup T_3 \cup T_4$. The complement of A is φ-closed.

Note (1.2) [8],[10] : A non-empty set X connected to with 4-topologies, and any member of it topologies holds condition in definition (1.1) namely "φ-topological spaces ($\mathrm{Top}_\varphi.S$)" (X, T_φ), $\varphi = 1,2,3,4$. Every fundamental topological space (Top.Space) is satisfied by open set in a Top.Space.

Definition (1.3) [11] : If (X, T_φ) is a Top.Space, and $A \subseteq X$ is a non-empty set, then A is namely g-open if for all closed sets E that are partial of A the relation is satisfied: $E \subseteq A_{int}$. Where A_{int} is the interior of A.

Definition (1.4) : If (X, T_φ) is a $\mathrm{Top}_\varphi.S$, and a set $\emptyset \neq A \subseteq X$, then A is i-open if: $A \subseteq (A \cap R)_{cl}$, when $(A \cap R)_{cl}$ is closure of $A \cap R$, with any proper subset R of X.

Definition (1.5) : If (X, T_φ) is a $\mathrm{Top}_\varphi.S$, and a set $\emptyset \neq A \subseteq X$, then A is $iD^{\#}$-open ($open_{iD^{\#}}$) if: A subset of $T_{1(int)}\{i - T_{2(cl)}[T_{3(int)}\{i - T_{4(cl)}(A)\}]\}$, where $i - T_{1(int)}, i - T_{3(int)}$ are interior of T_1, and T_3 respectively, $i - T_{2(cl)}, i - T_{4(cl)}$ are closure of i-T_2, and i-T_4 respectively.

Definition (1.6) : If (X, T_φ) is a $\mathrm{Top}_\varphi.S$, and $A \subseteq X$ is a non-empty set, then A is $iD_g^{\#}$-open ($open_{iD_g^{\#}}$) if: for all $iD^{\#}$-closed sets E that are partial of A the relation is satisfied: $E \subseteq A_{int}$. Where A_{int} is the interior of $open_{iD^{\#}}$ A.

Example (1.7) : Let $X = \{a, r, f\}$, where $T_1 = \{X, \emptyset, \{a\}\}$, $T_2 = \{X, \emptyset, \{a, r\}\}$, $T_2^c = \{X, \emptyset, \{f\}\}$,$i - T_2 = \{X, \emptyset, \{a\}, \{r\}, \{a, r\}, \{a, f\}, \{r, f\}\}$, i- $T_2^c = \{X, \emptyset, \{r, f\}, \{a, f\}, \{f\}, \{r\}, \{a\}\}$, $T_3 = \{X, \emptyset, \{f\}\}$, $T_4 = \{X, \emptyset, \{a, f\}\}$, $T_4^c = \{X, \emptyset, \{r\}\}$, i- $T_4 = \{X, \emptyset, \{a\}, \{f\}, \{a, f\}, \{a, r\}, \{r, f\}\}$, i- $T_4^c = \{X, \emptyset, \{r, f\}, \{a, r\}, \{r\}, \{f\}, \{a\}\}$. Then (X, T_φ) is a $T_\varphi.S$. So $open_{iD^{\#}}(X) = \{X, \emptyset, \{a, f\}\}$. Thus, the set of all $iD^{\#}$-closed set in X is $closed_{iD^{\#}}(X) = \{X, \emptyset, \{r\}\}$. Finally $open_{iD_g^{\#}}(X) = \{X, \emptyset, \{a\}, \{f\}, \{a, f\}\}$.

Example (1.8) : Let $X = \{a, t, d\}$, where $T_1 = \{X, \emptyset, \{a\}, \{t\}, \{a, t\}\}$, $T_2 = \{X, \emptyset, \{d\}\}$, $T_2^c = \{X, \emptyset, \{a, t\}\}$,$i - T_2 = \{X, \emptyset, \{t\}, \{a, t\}, \{a\}\}$, i- $T_2^c = \{X, \emptyset, \{a, d\}, \{d\}, \{t, d\}\}$, $T_3 = \{X, \emptyset, \{t\}, \{a, t\}$, $T_4 = \{X, \emptyset, \{a, t\}\}$, $T_4^c = \{X, \emptyset, \{d\}\}$,$i$- $T_4 = \{X, \emptyset, \{a\}, \{t\}, \{a, t\}, \{a, d\}, \{t, d\}\}$, i- $T_4^c = \{X, \emptyset, \{t, d\}, \{a, t\}, \{d\}, \{t\}, \{a\}\}$. Then (X, T_φ) is a $T_\varphi.S$. So $open_{iD^{\#}}(X) = \{X, \emptyset, \{t\}, \{a, t\}\}$. Thus, $closed_{iD^{\#}}(X) = \{X, \emptyset, \{a, d\}, \{d\}\}$. Finally $open_{iD_g^{\#}}(X) = \{X, \emptyset, \{a\}, \{t\}, \{a, t\}\}$, consequently $open_{iD_g^{\#}}(X)$ is Top.Space.

2. Maximal $open_{iD_g^{\#}}$

This section covers the basic concepts of maximal $open_{iD_g^\#}$ sets in quad topological spaces. These concepts form the basis for the mathematical analysis of these sets, including basic definitions and elementary properties. Application examples of these concepts are presented.

Definition (2.1) : An $open_{iD_g^\#}$ set $\emptyset \neq$ A of a $\mathrm{Top}_\varphi.S$ is claimed to be a maximal $open_{iD_g^\#}$ set ($iD_g^\#{}_{Max}$) iff any $open_{iD_g^\#}$ set which is may be found in A is X or A.

Example (2.2): Let $X = \{a, s, z\}$, $open_{iD_g^\#}(X) = \{X, \emptyset, \{a\}, \{z\}, \{a, z\}\}$. Then $\{a, z\}$ is maximal $iD_g^\#{}_{Max}$-open sets.

Lemma (2.3): Let D, and V are a $iD_g^\#{}_{Max}$-open sets with M an $open_{iD_g^\#}$ set. Then

1. $M = X$, or $M \subset D$.
2. $D \cup V = X$, or $D = V$.

Proof: 1. Take M is an $open_{iD_g^\#}$ set, and $D \cup M \neq X$, Given that D $iD_g^\#{}_{Max}$-open sets, and $D \subset D \cup M \Longrightarrow D \cap M = D$, therefore $M \subset D$

2. take $D \cap V \neq X \Longrightarrow$ either $D \subset V$ or $V \subset D$. From (1) we get $D = V$.

Proposition(2.4): Let F be a $iD_g^\#{}_{Max}$-open sets. If a is an element of F, then F subset of W, as W is any $open_{iD_g^\#}$-neighbourhood of x, $F \cup W = X$, or $F = W$

Proof: directly by [Lemma(2.3)].

Proposition (2.5): take U_α, U_β, and U_δ are a $iD_g^\#{}_{Max}$-open sets, with $U_\alpha \neq U_\beta$, if $U_\alpha \cap U_\beta \subset U_\delta$, then either $U_\beta \subset U_\delta$, or $U_\alpha \subset U_\delta$.

Proof: we see

$U_\alpha \cap U_\delta = U_\alpha \cap (U_\delta \cap X)$

$U_\alpha \cap U_\delta = U_\alpha \cap \left(U_\delta \cap \{U_\alpha \cup U_\beta\}\right)$ from **[Lemma (2.3)]**

$U_\alpha \cap U_\delta = U_\alpha \cap \left((U_\delta \cap U_\alpha) \cup (U_\delta \cap U_\beta)\right)$

$U_\alpha \cap U_\delta = (U_\alpha \cap U_\delta) \cup \left(U_\alpha \cap U_\beta \cap U_\delta\right)$

$U_\alpha \cap U_\delta = U_\alpha \cup \left(U_\alpha \cap U_\beta\right)$

If $U_\delta \neq U_\beta \Longrightarrow U_\beta \cup U_\delta = X$, and $U_\alpha \cap U_\delta = U_\alpha$, consequently $U_\alpha \subset U_\delta$

Then Given that U_α, U_β are a $iD_g^\#{}_{Max}$-open sets $\Longrightarrow U_\alpha = U_\delta$

Proposition (2.6) : Let U be a $iD_g^\#{}_{Max}$-open sets and $x \in U$. Then,

$U = \cup\{W \mid W\ open_{iD_g^\#}$-neighbourhood of x as $W \cup U \neq X\}$

Proof From [**Theorem (2.4)**] and by assuming U is $open_{iD_g^\#}$-neighbourhood of x. We get:

$U \subset \cup\{W \mid W\ open_{iD_g^\#}$-neighbourhood of x as $W \cup U \neq X\} \subset U$.

Finally, the presumptions hold

Theorem (2.7): for any finite $open_{iD_g^\#}$ set $V \subset X$, there exist at least one finite $iD_g^\#{}_{Max}$-open U as $V \subset U$

Proof if V is a $iD_g^\#{}_{Max}$-open set, we may set $U = V$. If V is not a $iD_g^\#{}_{Max}$-open set, then there exists an finite $open_{iD_g^\#}$ set V_1 as $V \subsetneq V_1 \neq X$. If V_1 is a $iD_g^\#{}_{Max}$-open set, we may set $U = V_1$. If V_1 is not a $iD_g^\#{}_{Max}$-open set, then there exists an finite $open_{iD_g^\#}$ set V_1 as $V \subsetneq V_1 \subsetneq V_2 \neq X$. As we go, we are left with a series of $iD_g^\#{}_{Max}$-open set

$$V \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_k \subsetneq \cdots$$

Given that V is a finite $open_{iD_g^\#}$ set, this process repeats only finitely. Thus, finally, then have get get a $iD_g^\#{}_{Max}$-open set $U = V_n$ for some $n \in \mathbb{Z}^+$.

3. Closure, interior, and $iD^{\#}_{g\,Max}$-open sets.

Theorem (3.1): to any $iD^{\#}_{g\,Max}$-open U, with element $x \in X - U$. Then, $X - U$ proper sub set of W $open_{iD^{\#}_g}$-neighbourhood of x.

Proof by presumption $x \in X - U \implies W \not\subset$ U, for of W $open_{iD^{\#}_g}$-neighbourhood of x. Then, From [**Lemma (2.3)**] $W \cup U = X$. Therefore, X^c proper sub set of W.

Corollary (3.2): Let any $iD^{\#}_{g\,Max}$-open U. Then, either for every $x \in X - U$, and W $open_{iD^{\#}_g}$-neighbourhood of x , $W = X$, or $\exists$ $open_{iD^{\#}_g}$ set W, as X^c proper sub set of W, and W proper sub set of X.

Proof. If for every $x \in X - U$, and W $open_{iD^{\#}_g}$-neighbourhood of x , $W = X$ not hold, then $\exists$ $x \in X - U$, and W $open_{iD^{\#}_g}$-neighbourhood of x as W proper sub set of X from [**Theorem (3.1)**] $X - U$ proper sub set of W.

Corollary (3.3): Let any $iD^{\#}_{g\,Max}$-open U. Then, either for every $\in X - U$, and W $open_{iD^{\#}_g}$-neighbourhood of x , then have get $X - U \subsetneq W$, or $\exists$ $open_{iD^{\#}_g}$ set W, as $X - U = W \neq X$.

Proof. If $\exists$ $open_{iD^{\#}_g}$ set W, as $X - U = W \neq X$ not hold, then from [**Theorem (3.1)**], then have get $X - U$ proper sub set of W, foe any $x \in X - U$, and W $open_{iD^{\#}_g}$-neighbourhood of x, consequently $X - U \subsetneq W$.

Theorem (3.4) : To any $iD^{\#}_{g\,Max}$-open U. Then, $U_{cl} = X$, or $U_{cl} = U$.

Proof. Given that U is a $iD^{\#}_{g\,Max}$-open set, from [**Corollary (3.3)**] one of presumption only hold:

(1) $\forall x \in X - U$, and $\forall$ W $open_{iD^{\#}_g}$-neighbourhood of x, get $X - U \subsetneq W$.

let $x \in X - U$ and W $open_{iD^{\#}_g}$-neighbourhood of x.

Given that $X/_U \neq W$, then have get $W \cap U \neq \emptyset, \forall$ W $open_{iD^{\#}_g}$-neighbourhood of x

Consequently, $X/_U \subset U_{cl}$. Given that $X = U \cup (X - U) \subset U \cup U_{cl} = U_{cl} \subset X$, then have get $U_{cl} = X$;

(2) $\exists$ W as $X/_U = W \neq X$: Given that $X - U = W$ is an $open_{iD^{\#}_g}$ set, U is a $closed_{iD^{\#}_g}$ set. Therefore, $U_{cl} = U$.

Theorem (3.5) : to any $iD^{\#}_{g\,Max}$-open U. Thus, $(X - U)_{int} = X$, or $(X - U)_{int} = \emptyset$.

Proof. Given that U is a $iD^{\#}_{g\,Max}$-open set, from [**Corollary (3.3)**] one of presumption only hold, then have get $(X - U)_{int} = X$, or $(X - U)_{int} = \emptyset$.

Theorem (3.6) : To any $iD^{\#}_{g\,Max}$-open set U and any $open_{iD^{\#}_g}$ sets, when $\emptyset \neq S$ of a $X - U$. Then $S_{iD^{\#}_{g\,cl}} = X - U$.

Proof Given that $\emptyset \neq S$ of a $X - U$, then have get $W \cap S \neq \emptyset$, for $x \in X - U$, and $\forall$ W $open_{iD^{\#}_g}$-neighborhood of x, from [**Theorem (3.1)**]. Then $X - U \subset S_{cl}$ {Given that $X - U$ is $closed_{iD^{\#}_g}$ set}, and $S \subset X - U$, we observe that $S_{cl} \subset (X - U)_{cl} = X - U$. Consequently $X - U = S_{iD^{\#}_{g\,cl}}$.

Corollary (3.7) : Let any $iD^{\#}_{g\,Max}$-open U and for any $open_{iD^{\#}_g}$ set, and $\emptyset \neq M \subset X$, with $U \subsetneq M$ Then $M_{cl} = X$.

Proof Given that $U \subsetneq M \subset X, \emptyset \neq S \subset X - U$ as $M = U \cup S$. Consequently, then have get $M_{cl} = (S \cup U)_{cl} = S_{cl} \cup U_{cl} \supset (X - U) \cup U = X$ from [**Theorem (3.6)**] Therefore, $M_{cl} = X$.

Theorem (3.8) : Let any $iD^{\#}_{g\,Max}$-open U and suppose $X - U$ has at least two elements. Then $(X - \{a\})_{cl} = X, \forall a \in X - U$.

Proof Given that $U \subsetneq X - \{a\}$ by presumption, then have get the outcome by [**Corollary (3.7)**]

Theorem (3.9) : To any $iD^{\#}_{g\,Max}$-open U and suppose N a proper $open_{iD^{\#}_g}$ subset of X with $U \subset N$. Thus, $N_{int} = U$.
Proof. If $N = U$, thus $N_{iD^{\#}_{g\,int}} = U_{int} = U$. Otherwise $N \neq U$, and consequently $U \subsetneq N$. It suggests that $U \subset N_{int}$. Given that U is a $iD^{\#}_{g\,Max}$-open set, then have get also $N_{int} \subset U$. Therefore, $N_{iD^{\#}_{g\,int}} = U$.
Theorem (3.10) : To any $iD^{\#}_{g\,Max}$-open U and suppose $\emptyset \neq S$ a proper $open_{iD^{\#}_g}$ subset of $X - U$. Thus, $X - S_{iD^{\#}_{g\,cl}} = (X - U\,)_{iD^{\#}_{g\,int}}$.
Proof Given that $U \subset X - S \subsetneq X$ by presumption, then have get the outcome by [**Theorem (3.6) and Theorem (3.9)**].
Definition(3.11) : A $open_{iD^{\#}_g}$ subset M in $\mathrm{T}_{\varphi}.S\ \left(X, \mathrm{T}_{\varphi}\right)$ is namely pre-$open_{iD^{\#}_g}$ ($preopen_{iD^{\#}_g}$) if $M \subset cl_{iD^{\#}_g}(M_{int})_{iD^{\#}_g}$.
Theorem (3.12) : Let any $iD^{\#}_{g\,Max}$-open U and any $open_{iD^{\#}_g}$ subset M of X, with $U \subset M$, then M is $preopen_{iD^{\#}_g}$ set.
Proof If $M = U$, then M is $open_{iD^{\#}_g}$ set. Therefore, M is a $preopen_{iD^{\#}_g}$ set. Otherwise, $U \subsetneq M$, then $int_{iD^{\#}_g}\left(M_{cl_{iD^{\#}_g}}\right) = X_{iD^{\#}_{g\,int}} = X \supset M$ by [**Corollary (3.7)**]. Consequently, M is a $preopen_{iD^{\#}_g}$ set.
Corollary(3.13) : To any $iD^{\#}_{g\,Max}$-open U. Then, $X - \{a\}$ is a $preopen_{iD^{\#}_g}$ set to any $a \in X - U$.
Proof Given that $U \subset X - \{a\}$ by presumption, from [**Theorem (3.11)**] then have get the outcome.

4. Fundamental properties of radicals

This section focused on the topological properties of $iD^{\#}_{g\,Max}$-open by studying the concepts of interior and radical closure.
Definition (4.1) : to any $iD^{\#}_{g\,Max}$-open $U_{\alpha}, \alpha \in \Lambda$, the intersection of all U_{α} is called radical of $U_{\alpha}, \alpha \in \Lambda$.
Theorem (4.2) : Let $|\Lambda| \geq 2$. any $iD^{\#}_{g\,Max}$-open $U_{\alpha}, \alpha \in \Lambda$ and $U_{\beta} \neq U_{\gamma}, \gamma, \beta \in \Lambda, \gamma \neq \beta$.
(1) for any $\gamma \in \Lambda$, $X \bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \subset U_{\gamma}$.
(2) for any $\gamma \in \Lambda$, $\bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \neq \emptyset$.
Proof (1) for any $\gamma \in \Lambda$. From [**Lemma 2.2**], then have get $X/U_{\gamma} \subset U_{\beta}$, for any $\beta \in \Lambda$, with $\gamma \neq \beta$. Then, $X/U_{\gamma} \subset \bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta}$. Therefore, then have get $X \bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \subset U_{\gamma}$.
(2) If $\bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \neq \emptyset$,then have get $X = U_{\gamma}$ from (1). X = Uμ by (1). This conflict presumption U_{γ} is $iD^{\#}_{g\,Max}$-open set. Then then have get $\bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \neq \emptyset$.
Corollary (4.3) : Any $iD^{\#}_{g\,Max}$-open $U_{\alpha}, \alpha \in \Lambda$ and $U_{\beta} \neq U_{\gamma}, \gamma, \beta \in \Lambda, \gamma \neq \beta$, if $|\Lambda| \geq 3$, then then $U_{\beta} \cap U_{\gamma} \neq \emptyset, \gamma, \beta \in \Lambda, \gamma \neq \beta$.
Proof By [**Theorem (4.2)**], then have get $U_{\beta} \cap U_{\gamma} \neq \emptyset, \gamma, \beta \in \Lambda, \gamma \neq \beta$.
Theorem(4.4) : Any $iD^{\#}_{g\,Max}$-open $U_{\alpha}, \alpha \in \Lambda$ and $U_{\beta} \neq U_{\gamma}, \gamma, \beta \in \Lambda, \gamma \neq \beta$, if $|\Lambda| \geq 2$, then then $U_{\beta} \cap U_{\gamma} \neq \emptyset, \gamma, \beta \in \Lambda, \gamma \neq \beta$. Then, $\bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \not\subset U_{\gamma} \not\subset \bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta}$.
Proof If $\bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \not\subset U_{\gamma}, \gamma \in \Lambda$, then we show $X = \left(X/\bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta}\right) \cup \bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta} \subset U_{\gamma}$, from [**Theorem (4.2)**]. This conflict presumption. If $U_{\gamma} \subset \bigcap_{\beta \in \Lambda/\{\gamma\}} U_{\beta}$, then $U_{\gamma} \subset U_{\beta}$, consequently $U_{\gamma} \subset U_{\beta}$, if $\beta \in \Lambda/\gamma$. This conflict presumption $U_{\gamma} \neq U_{\beta}$, when $\gamma \neq \beta$.
Corollary(4.5) : Any $iD^{\#}_{g\,Max}$-open $U_{\alpha}, \alpha \in \Lambda$ and $U_{\beta} \neq U_{\gamma}, \gamma, \beta \in \Lambda, \gamma \neq \beta$. If $\emptyset \neq \Gamma$ is a $open_{iD^{\#}_g}$-subset of Λ, thus $\bigcap_{\beta \in \Lambda/\Gamma} U_{\beta} \not\subset \bigcap_{\theta \in \Gamma} U_{\theta} \not\subset \bigcap_{\beta \in \Lambda/\Gamma} U_{\beta}$.

Proof Take $\theta \in \Gamma$, we show $\bigcap_{\beta\in\Lambda/\Gamma} U_\beta = \bigcap_{\beta\in(\{\Lambda/\Gamma\}\cup\{\theta\})/\{\theta\}} U_\beta \not\subset U_\theta$, from [**Theorem (4.4)**]. Therefore we show $\bigcap_{\beta\in\Lambda/\Gamma} U_\beta \not\subset \bigcap_{\theta\in\Gamma} U_\theta$. In other side Given that $\bigcap_{\theta\in\Gamma} U_\theta = \bigcap_{\theta\in\Gamma/(\Lambda/\Gamma)} U_\theta \not\subset \bigcap_{\beta\in\Gamma} U_\theta \not\subset \bigcap_{\beta\in\Lambda/\Gamma} U_\beta$.

Theorem (4.6): Any $iD^{\#}_{g\,Max}$-open $U_\beta, \beta \in\Lambda$ and $U_\beta \neq U_\gamma, \gamma, \beta \in\Lambda, \gamma \neq \beta$. If $\emptyset \neq \Gamma$ is a $open_{iD^{\#}_g}$-subset of Λ, then $\bigcap_{\beta\in\Lambda} U_\beta \subsetneq \bigcap_{\gamma\in\Gamma} U_\gamma$.

Proof By [**Corollary (4.5)**], then have get $\bigcap_{\beta\in\Lambda} U_\beta = \bigcap_{\beta\in\Lambda/\Gamma} U_\beta \cap \bigcap_{\beta\in\Gamma} U_\beta \subsetneq \bigcap_{\gamma\in\Gamma} U_\gamma$.

Theorem (4.7) : (a theorem of decomposition for $iD^{\#}_{g\,Max}$-open). Assume that $|\Lambda| \geq 2$. any $iD^{\#}_{g\,Max}$-open $U_\beta, \beta \in\Lambda$ and $U_\beta \neq U_\gamma, \gamma, \beta \in\Lambda, \gamma \neq \beta$. If $\emptyset \neq \Gamma$ is a $open_{iD^{\#}_g}$-subset of Λ, thus, for $\gamma \in\Lambda$, $U_\gamma = \bigcap_{\gamma\in\Lambda} U_\gamma \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta$.

Proof For $\gamma \in\Lambda$. From [**Theorem (4.2)**], then have get

$\bigcap_{\beta\in\Lambda} U_\beta \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta = \bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta \cap U_\gamma \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta$

$\bigcap_{\beta\in\Lambda} U_\beta \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta = \bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta \cap U_\gamma \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta$

$\bigcap_{\beta\in\Lambda} U_\beta \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta = U_\gamma \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta$

$\bigcap_{\beta\in\Lambda} U_\beta \cup X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta = U_\gamma$

Therefore, then have get $U_\gamma = \left(\bigcap_{\beta\in\Lambda} U_\beta\right) \cup \left(X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta\right)$.

Theorem (4.8) : Any $iD^{\#}_{g\,Max}$-open $U_\beta, \beta \in\Lambda$, where a finite Λ, and $U_\beta \neq U_\gamma, \gamma, \beta$. If $\bigcap_{\beta\in\Lambda} U_\beta$ is a $closed_{iD^{\#}_g}$ set, then $U_\beta, \beta \in\Lambda$ is $closed_{iD^{\#}_g}$.

Proof from [**Theorem (4.7)**], then have get

$U_\gamma = \left(\bigcap_{\beta\in\Lambda} U_\beta\right) \cup \left(X/\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta\right)$

$U_\gamma = \left(\bigcap_{\beta\in\Lambda} U_\beta\right) \cup \left(\bigcup_{\beta\in\Lambda/\{\gamma\}} X/U_\beta\right)$

Given that Λ is a finite set, we see $\bigcup_{\beta\in\Lambda/\{\gamma\}} X/U_\beta$ is a $closed_{iD^{\#}_g}$ set. Consequently, U_γ is a $closed_{iD^{\#}_g}$ set from presumption . As an use of [**Theorem (4.7)**].

Theorem (4.9) : For $|\Lambda| \geq 2$ any $iD^{\#}_{g\,Max}$-open $U_\beta, \beta \in\Lambda$, where a finite Λ, and $U_\beta \neq U_\gamma, \gamma, \beta$. If $\bigcap_{\beta\in\Lambda} U_\beta = \emptyset$, then $\{U_\beta, \beta \in\Lambda\}$ is this set of all $iD^{\#}_{g\,Max}$-open sets of X.

Proof If $\exists$an additional $iD^{\#}_{g\,Max}$-open set U_δ of X, $U_\beta \neq U_\delta$, $\beta \in\Lambda$, then $\emptyset \neq \bigcap_{\beta\in\Lambda} U_\beta = \bigcap_{\beta\in\Lambda\cup\{\delta\}/\{\delta\}} U_\beta$. from [**Theorem (4.2)**], we observe that $\bigcap_{\beta\in\Lambda\cup\{\delta\}/\{\delta\}} U_\beta \neq \emptyset$. This conflict presumption.

Example (4.10) : If each point $\{x\}$ is $closed_{iD^{\#}_g}$ (e.g., X is a Hausdorff or a finite or a countable space), then $X/\{a\}$ is a $iD^{\#}_{g\,Max}$-open set for any $a \in X$. Moreover, we see $\{X/\{a\} \mid a \in X\}$is this set of all $iD^{\#}_{g\,Max}$-open sets of X from [**Theorem (4.9)**], Given that $\bigcap_{a\in X} X/\{a\} = \emptyset$.

Proposition (4.11) : For any $open_{iD^{\#}_g}$ $A, B \subset X$. If $A \cup B = X$, $A \cap B$ is a $closed_{iD^{\#}_g}$ set, and A is $open_{iD^{\#}_g}$ set, then B is $closed_{iD^{\#}_g}$ set.

Proof Given that $X/A \subset B$, then we observe

$(A \cap B) \cup X/A = A \cup X/A \cap B \cup (X/A = B \cup X/A = B$. (4.4) Given that $A \cap B$ and X/A are $closed_{iD^{\#}_g}$ sets, we observe that B is a $closed_{iD^{\#}_g}$ set.

Proposition (4.12) : If $U_\beta, \beta \in\Lambda$ is an $open_{iD^{\#}_g}$ and $U_\beta \cup U_\gamma = X, \beta, \gamma \in\Lambda$, $\beta \neq \gamma$. If $\bigcap_{\beta\in\Lambda} U_\beta$ is a $closed_{iD^{\#}_g}$ set, then $\bigcap_{\beta\in\Lambda\cup\{\gamma\}/\{\gamma\}} U_\beta$ is a $closed_{iD^{\#}_g}$ set.

Proof If $\gamma \in\Lambda$. Given that $U_\beta \cup U_\gamma = X, \beta, \gamma \in\Lambda$, $\beta \neq \gamma$, then have get $U_\gamma \cup \bigcap_{\beta\in\Lambda\cup\{\gamma\}/\{\gamma\}} U_\beta = \bigcap_{\beta\in\Lambda\cup\{\gamma\}/\{\gamma\}} U_\beta \cup U_\gamma = X$. Given that $U_\gamma \cup \bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta = \bigcap_{\beta\in\Lambda} U_\beta$ is a $closed_{iD^{\#}_g}$ set by presumption, $\bigcap_{\beta\in\Lambda\cup\{\gamma\}/\{\gamma\}} U_\beta$is a $closed_{iD^{\#}_g}$ set from [**Prop. (4.11)**].

Theorem(4.13) : Any $iD^{\#}_{g\,Max}$-open $U_\beta, \beta \in\Lambda$, where a finite Λ, and $U_\beta \neq U_\gamma, \gamma, \beta$. If $\bigcap_{\beta\in\Lambda} U_\beta$ is a $closed_{iD^{\#}_g}$ set, then$\bigcap_{\beta\in\Lambda/\{\gamma\}} U_\beta$ is a $closed_{iD^{\#}_g}$ set.

Proof from [**Lemma (2.2)**], then have get $U_\beta \cup U_\gamma = X, \beta, \gamma \in \wedge$, $\beta \neq \gamma$. By [**Prop. (4.12)**], then have get $\bigcap_{\beta\in\wedge/\{\gamma\}} U_\beta$ is a $closed_{iD_g^\#}$ set. If the presumption of [**Prop. (4.12)**] not maintain, then condition that $\bigcap_{\beta\in\wedge} U_\beta$ is a $closed_{iD_g^\#}$ set $\nRightarrow \bigcap_{\beta\in\wedge/\{\gamma\}} U_\beta$ is $closed_{iD_g^\#}$ set. Here's an example.

Example (4.14): Let $X = \{a, b, c\}$, where $open_{iD_g^\#}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, thus $\{a\} \cap \{b\} \cap \{a, b\} = \emptyset$, is $closed_{iD_g^\#}$ set , $\{a\} \cup \{b\} \cup \{a, b\} = \{a, b\} \neq X$ is $closed_{iD_g^\#}$ set, $\{a\} \cup \{b\} = \{a, b\} \neq X$ is $closed_{iD_g^\#}$ set.

5. More radicals of $iD_{g_{Max}}^\#$-open sets

Proposition (5.1): Let any $iD_{g_{Max}}^\#$-open $U_\beta, \beta \in \wedge$. If $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = X$, thus $U_{\beta_{cl}} = X$ $\beta \in \wedge$.

Proof. We observe $X = \left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} \subset U_{\beta_{cl}}$. It suggests that $U_{\beta_{cl}} = X$ $\beta \in \wedge$.

Theorem (5.2) : Any $iD_{g_{Max}}^\#$-open $U_\beta, \beta \in \wedge$. If $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} \neq X$, then $\exists$ an element $\beta \in \wedge$ such $U_{\beta_{cl}} = U_\beta$.

Proof Assuming that $U_{\beta_{cl}} = X$ $\beta \in \wedge$.. Let $\delta \in \wedge$ Given that $\bigcap_{\beta\in\wedge/\{\delta\}} U_\beta$ is $open_{iD_g^\#}$ set, then have get $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \bigcap_{\beta\in\wedge/\{\delta\}} U_\beta \cap U_\delta \supset \bigcap_{\beta\in\wedge/\{\delta\}} U_\beta \cap U_{\delta_{cl}}$

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \bigcap_{\beta\in\wedge/\{\delta\}} U_\beta \cap X$

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \bigcap_{\beta\in\wedge/\{\delta\}} U_\beta$

Consequently, $\left(\bigcap_{\beta\in\wedge/\{\delta\}} U_\beta\right)_{cl} \subset \left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl}$.

On the other hand, we observe that $\bigcap_{\beta\in\wedge} U_\beta \subset \bigcap_{\beta\in\wedge/\{\delta\}} U_\beta$, and consequently $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} \subset \left(\bigcap_{\beta\in\wedge/\{\delta\}} U_\beta\right)_{cl}$. It suggests that $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \left(\bigcap_{\beta\in\wedge/\{\delta\}} U_\beta\right)_{cl}$

By continuously in same way then get for any element in $\wedge$, then see $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = U_{\beta_{cl}} = X$. This conflict presumption. that $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} \neq X$. Therefore, then observe that $\exists$ an element $\beta \in \wedge$. such that $U_{\beta_{cl}} = U_\beta$.

Theorem (5.3) : (the law of radical closure). L any $iD_{g_{Max}}^\#$-open $U_\beta, \beta \in \wedge$, where a finite $\wedge$. take Γ be a $open_{iD_g^\#}$ subset of $\wedge$ so that $U_{\beta_{cl}} = U_\beta$ to any $\beta \in \Gamma$, $U_{\beta_{cl}} = X$ for any $\beta \in \wedge/\Gamma$. Then, $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \bigcap_{\beta\in\Gamma} U_\beta$, or $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = X$ if $\Gamma = \emptyset$.

Proof If $\Gamma = \emptyset$, then then have get the outcome from [**Theorem (5.2)**]. If $\Gamma \neq \emptyset$, and consequently we observe that :

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \left(\left(\bigcap_{\beta\in\Gamma} U_\beta\right) \cap \left(\bigcap_{\beta\in\wedge/\Gamma} U_\beta\right)\right)_{cl} \supset \bigcap_{\beta\in\Gamma} U_\beta \cap \bigcap_{\beta\in\wedge/\Gamma} U_\beta$

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \bigcap_{\beta\in\Gamma} U_\beta \cap X$

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \bigcap_{\beta\in\Gamma} U_\beta$, the by [**Theorem (5.2)**] and the $\bigcap_{\beta\in\Gamma} U_\beta$ is $open_{iD_g^\#}$ set. It suggests that

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \left(\left(\bigcap_{\beta\in\Gamma} U_\beta\right)_{cl}\right)_{cl} \supset \left(\bigcap_{\beta\in\Gamma} U_\beta\right)_{cl}$. On the other hand, we observe that $\bigcap_{\beta\in\wedge} U_\beta \subset \bigcap_{\beta\in\Gamma} U_\beta$, and consequently $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} \subset \left(\bigcap_{\beta\in\Gamma} U_\beta\right)_{cl}$. It suggests that

$\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \left(\bigcap_{\beta\in\Gamma} U_\beta\right)_{cl}$. The radical $\bigcap_{\beta\in\Gamma} U_\beta$ is $closed_{iD_g^\#}$ set Given that U_β is a $closed_{iD_g^\#}$,set for any $\lambda\beta \in \Gamma$ by presumption. Therefore, we observe that $\left(\bigcap_{\beta\in\wedge} U_\beta\right)_{cl} = \left(\bigcap_{\beta\in\Gamma} U_\beta\right)_{cl}$.

Conclusion: $iD_{g_{Max}}^\#$-open is a type open sets represent a new addition to the understanding of topological spaces. The outcomes can be used to enhance the study of spaces with complex structures such as pre-Hausdorff spaces. The research opens new horizons for the development of advanced analytic theories that can be applied in topology and pure mathematics.

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النظريات الأساسية وبعض خصائص المجموعات المفتوحة العظمى من النمط $iD_g^\#$ في الفضاءات الطوبولوجية الرباعية

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الكلمات المفتاحية:
طوبولوجيا رباعية، مجموعة مفتوحة من النمط $iD_g^\#$، مجموعة عظمى مفتوحة من النمط $iD_g^\#$ ، إغلاق جذري، ونظرية التحلل.

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الخلاصة:
تتناول هذه الدراسة الخصائص الأساسية للمجموعات المفتوحة عظمى من النمط $iD_g^\#$ في مجال الطوبولوجيا. ويركز البحث على مجموعة من النتائج النظرية المهمة، بما في ذلك إثبات نظرية التحلل، التي تشرح كيف يمكن تقسيم هذه المجموعات إلى مجموعات فرعية أبسط مع الحفاظ على خصائصها الجوهرية. بالإضافة إلى ذلك، تبحث الدراسة في الخصائص المرتبطة بتقاطع المجموعات المفتوحة عظمى من النمط $iD_g^\#$، مع التركيز على العلاقة بين هذه التقاطعات وانخفاض الإغلاق الجذري. يتم توفير أدلة دقيقة لهذه الخصائص وإثبات أهميتها النظرية في فهم ديناميكيات المجموعات المفتوحة في هذا النوع. تساهم النتائج في الفهم الرياضي للطوبولوجيا من خلال توفير طرق تحليل جديدة يمكن تطبيقها على دراسة فئة جديدة من المجموعات المفتوحة. ستفتح النتائج المقدمة آفاقًا جديدة للتطبيقات النظرية في الرياضيات البحتة، وخاصة في الطوبولوجيا العامة والتحليل الوظيفي.